

Kerr Metric

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Einstein Field Equations and the Simplest Solution

Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Vacuum Solution: $T_{\mu\nu} = 0$

$$R_{\mu\nu} = 0$$

Schwarzschild Metric:

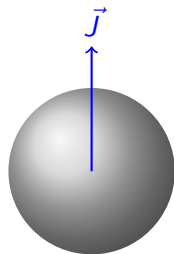
$$ds^2 = - \left(1 - \frac{2GM}{r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

- ▶ Static, spherically symmetric solution
- ▶ Describes spacetime outside a non-rotating, uncharged mass

Kerr Metric — Rotating Black Hole

Kerr Metric:

- ▶ Solution to Einstein's equations in vacuum ($T_{\mu\nu} = 0$)
- ▶ Describes the geometry around a **rotating, uncharged** black hole
- ▶ Introduces angular momentum into spacetime
- ▶ Spacetime becomes **axisymmetric** and **stationary**, not static



Key parameter:

$$a = \frac{J}{Mc} \quad (\text{specific angular momentum})$$

Step 1: Kerr–Schild Ansatz

Einstein Field Equations (general form):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Vacuum case: $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

Kerr's approach: use the Kerr–Schild ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_\mu l_\nu$$

- ▶ $\eta_{\mu\nu}$: flat Minkowski metric
- ▶ H : scalar function (depends on coordinates)
- ▶ l_μ : null vector in both $\eta_{\mu\nu}$ and $g_{\mu\nu}$

$$\eta^{\mu\nu} l_\mu l_\nu = 0, \quad g^{\mu\nu} l_\mu l_\nu = 0$$

Step 2: Derivation of Inverse Metric

Given the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_\mu l_\nu$$

where $\eta_{\mu\nu}$ is the Minkowski metric, H is a scalar function, and l_μ is a null vector:

$$\eta^{\mu\nu} l_\mu l_\nu = 0$$

Claim: The inverse metric is:

$$g^{\mu\nu} = \eta^{\mu\nu} - 2Hl^\mu l^\nu$$

Verification: Multiply to check $g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$

$$(\eta^{\mu\alpha} - 2Hl^\mu l^\alpha)(\eta_{\alpha\nu} + 2Hl_\alpha l_\nu)$$

$$= \delta_\nu^\mu + 2Hl^\mu l_\nu - 2Hl^\mu l_\nu - 4H^2 l^\mu (l^\alpha l_\alpha) l_\nu$$

Since $l^\alpha l_\alpha = 0$, the last term vanishes:

$$g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$$

Step 3: Einstein Equation with Metric Substitution

Einstein Field Equations (general form):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

In vacuum: $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

We substitute the Kerr–Schild metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_{\mu}l_{\nu}$$

Our goal: compute the Ricci tensor $R_{\mu\nu}$ using this form.

- ▶ All curvature comes from derivatives of H and l_{μ}
- ▶ Background $\eta_{\mu\nu}$ is flat $\Rightarrow R_{\mu\nu}[\eta] = 0$
- ▶ We will plug into the definition of the Ricci tensor and simplify

Next step: compute $\Gamma_{\mu\nu}^{\lambda}$ and derive an expression for $R_{\mu\nu}$

Step 4: Christoffel Symbol Calculation

Christoffel symbols:

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\sigma} (\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu})$$

Kerr–Schild metric: $g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_{\mu}l_{\nu}$

Compute derivative:

$$\partial_{\alpha}g_{\mu\nu} = 2(\partial_{\alpha}H)l_{\mu}l_{\nu} + 2H[(\partial_{\alpha}l_{\mu})l_{\nu} + l_{\mu}(\partial_{\alpha}l_{\nu})]$$

Substitute into Christoffel expression:

$$\Gamma_{\mu\nu}^{\lambda} = g^{\lambda\sigma} [(\partial_{\mu}H)l_{\sigma}l_{\nu} + (\partial_{\nu}H)l_{\sigma}l_{\mu} - (\partial_{\sigma}H)l_{\mu}l_{\nu}] + \text{terms with } \partial l$$

Now use $g^{\lambda\sigma} = \eta^{\lambda\sigma} - 2Hl^{\lambda}l^{\sigma}$ and l^{μ} null $\Rightarrow l^{\mu}l_{\mu} = 0$ to simplify:

$$\Gamma_{\mu\nu}^{\lambda} = \partial_{\mu}H l^{\lambda}l_{\nu} + \partial_{\nu}H l^{\lambda}l_{\mu} - \partial^{\lambda}H l_{\mu}l_{\nu} + \dots$$

The omitted terms involve derivatives of l_{μ} and vanish or simplify under geodesic assumptions.

Step 5: Ricci Tensor Derivation

Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda$$

Use:

$$\Gamma_{\mu\nu}^\lambda = \partial_\mu H l^\lambda l_\nu + \partial_\nu H l^\lambda l_\mu - \partial^\lambda H l_\mu l_\nu$$

First term:

$$\begin{aligned}\partial_\lambda \Gamma_{\mu\nu}^\lambda &= \partial_\lambda (\partial_\mu H l^\lambda l_\nu + \partial_\nu H l^\lambda l_\mu - \partial^\lambda H l_\mu l_\nu) \\ &= (\partial_\lambda \partial_\mu H) l^\lambda l_\nu + \partial_\mu H \partial_\lambda l^\lambda l_\nu + \dots\end{aligned}$$

Retaining only leading terms under null and geodesic assumptions:

$$\partial_\lambda \Gamma_{\mu\nu}^\lambda = (\partial_\lambda \partial^\lambda H) l_\mu l_\nu = \square H \cdot l_\mu l_\nu$$

Second term:

$$\begin{aligned}\partial_\nu \Gamma_{\mu\lambda}^\lambda &= \partial_\nu (\partial_\mu H l^\lambda l_\lambda + \partial_\lambda H l^\lambda l_\mu - \partial^\lambda H l_\mu l_\lambda) \\ &= 0 \quad (\text{since } l^\lambda l_\lambda = 0 \text{ and geodesic assumptions})\end{aligned}$$

Step 6: Coordinate Choice and Symmetry

We have reduced Einstein's equations to:

$$R_{\mu\nu} = -\square H \cdot l_\mu l_\nu \quad \Rightarrow \quad \boxed{\square H = 0}$$

To solve this equation, we exploit **physical symmetry**:

- ▶ We are looking for a rotating solution \rightarrow spacetime must be **stationary** and **axisymmetric**
- ▶ These symmetries are best captured in a generalized spherical coordinate system

We introduce **oblate spheroidal coordinates** (t, r, θ, ϕ) , adapted to rotation:

$$\begin{cases} x = \sqrt{r^2 + a^2} \sin \theta \cos \phi \\ y = \sqrt{r^2 + a^2} \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

In these coordinates, we will construct a null geodesic vector l^μ and solve $\square H = 0$

Step 7: Choosing the Null Vector l^μ

We now choose a vector l^μ that is:

- ▶ **Null:** $\eta_{\mu\nu} l^\mu l^\nu = 0$
- ▶ **Geodesic:** $l^\nu \partial_\nu l^\mu = 0$

In spheroidal coordinates (t, r, θ, ϕ) , we use:

$$l^\mu = \left(1, \frac{r^2 + a^2}{\Delta}, 0, \frac{a}{\Delta} \right)$$

With parameters:

$$\Delta = r^2 - 2Mr + a^2, \quad a = \frac{J}{Mc}$$

- ▶ Δ : Appears in radial functions and determines horizon structure
- ▶ a : **Specific angular momentum** — rotation per unit mass

This choice of l^μ satisfies both required conditions and leads to the correct structure of the Kerr spacetime.

Step 8: Solving the PDE $\square H = 0$ — Part 1

We solve the reduced field equation:

$$\square H = 0 \quad (\text{in flat space})$$

Assume axial symmetry: $H = H(r, \theta)$

In oblate spheroidal coordinates, define:

$$\Sigma = r^2 + a^2 \cos^2 \theta$$

Laplacian in flat background:

$$\square H = \frac{\partial^2 H}{\partial r^2} + \frac{2r}{\Sigma} \frac{\partial H}{\partial r} + \frac{1}{\Sigma} \frac{\partial^2 H}{\partial \theta^2} + \frac{\cot \theta}{\Sigma} \frac{\partial H}{\partial \theta}$$

We use the ansatz:

$$H(r, \theta) = \frac{f(r)}{\Sigma}$$

Next, we compute derivatives of this function and plug them into

$$\square H$$

Step 9: Assembling the Kerr Metric – Kerr–Schild Form

We have constructed the Kerr metric using the Kerr–Schild ansatz:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_{\mu}l_{\nu}$$

With:

$$H = \frac{Mr}{r^2 + a^2 \cos^2 \theta} \quad , \quad l^{\mu} = \left(1, \frac{r^2 + a^2}{\Delta}, 0, \frac{a}{\Delta}\right)$$

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta$$

This combination satisfies:

$$R_{\mu\nu} = 0$$

and describes a rotating black hole in vacuum.

Step 10: Final Result – Kerr Metric (Boyer–Lindquist Form)

The Kerr metric in Boyer–Lindquist coordinates (t, r, θ, ϕ) is:

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi \\ & + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ & + \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 \end{aligned}$$

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2$$

This is the exact solution for an uncharged rotating black hole in general relativity.