Quantum Gravity

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linearized gravity

The field equations of linearized gravity can be obtained from the following Lagrangian:
(Fierz and Pauli 1939):

$$\mathcal{L} = \frac{1}{64\pi G}\left(f^{\mu\nu,\sigma} f_{\mu\nu,\sigma} - f^{\mu\nu,\sigma} f_{\sigma\nu,\mu} - f^{\nu\mu,\sigma} f_{\sigma\mu,\nu} - f^{\mu}{}_{\mu,\nu} f^{\rho\nu}{}_{,\rho} + 2 f^{\rho\nu}{}_{,\nu} f^{\sigma}{}_{\sigma,\rho}\right) - \frac{1}{2} T_{\mu\nu} f^{\mu\nu}$$

Where : $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$

The Euler–Lagrange field equations are:

$$f^{\sigma}_{\mu\nu,\sigma} - f^{\sigma}_{\sigma\mu,\nu} - f^{\sigma}_{\sigma\nu,\mu} + f_{,\mu\nu} + \eta_{\mu\nu}\left(f^{\alpha\beta}_{,\alpha\beta} - f_{,\sigma\sigma}\right) = -16\pi G T_{\mu\nu}$$

Using ‘harmonic condition’ (‘de Donder gauge’) $f^{\nu}_{\mu\nu} = \frac{1}{2} f^{\nu}_{\nu,\mu}$ we can derive Einstein equations in the linear approximation:

$$\Box f_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T\right)$$

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Gravitons from representations of the Poincaré group

Gravity is described by massless spin-2 particle called Graviton

Why massless?

- From the long-range nature of the gravitational interaction, it is clear that the graviton must have a small mass.
- Massive version of the Fierz–Pauli Lagrangian does not lead to linearized GR in the limit when the mass is set to zero.

strongest from gravitational effects on the scale of galaxy clusters, gives $m_g \lesssim 10^{-29}\,\mathrm{eV}$

Why spin-2?

According to Wigner, ‘particles’ are classified by irreducible representations of the Poincaré group.

Poincaré transformation is described as: $$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu},$$

According to Wigner’s theorem, Poincaré transformation induces a unitary transformation in the Hilbert space

$$\psi \to U(\Lambda, a)\psi$$

Since this is a Lie group, it is advantageous to study group elements close to the identity,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \quad a^{\mu} = \epsilon^{\mu},$$

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Gravitons from representations of the Poincaré group

This corresponds to the unitary transformation: $$U(1+\omega,\epsilon) = 1 + \frac{1}{2} i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu + \cdots ,$$

where $J^{\mu\nu}$ and P^μ denote the 10 Hermitian generators of the Poincaré group, which are the boost generators, the angular momentum and the four-momentum

Lie-algebra relations:

$$[P^\mu, P^\rho] = 0,$$
$$i[J^{\mu\nu}, J^{\lambda\rho}] = \eta^{\nu\lambda} J^{\mu\rho} - \eta^{\mu\lambda} J^{\nu\rho} - \eta^{\rho\mu} J^{\lambda\nu} + \eta^{\rho\nu} J^{\lambda\mu},$$
$$i[P^\mu, J^{\lambda\rho}] = \eta^{\mu\lambda} P^\rho - \eta^{\mu\rho} P^\lambda$$

Since the components P^μ of the four-momentum commute with each other, we choose their eigenstates

$$P^u \psi_{p,\sigma} = p^u \psi_{p,\sigma}$$

Application of the unitary operator then yields $$U(1,a)\psi_{p,\sigma} = e^{-ip^\mu a_\mu} \psi_{p,\sigma}$$

Since One-particle states are classified according to their behaviour with respect to Poincaré transformations we need to find how these states transform under Lorentz transformations

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Gravitons from representations of the Poincaré group

According to the method of induced representations, it is sufficient to find the representations of the little group. This group is characterized by the fact that it leaves a 'standard' vector k^μ invariant (within each class of given$p^2 \leq 0$ and given sign of p^0). For positive p^0 , one can distinguish the following two cases.

- The first possibility is p^2 = − m^2 < 0. Here one can choose

k^μ = (m, 0, 0, 0), and the little group is SO(3), since these are the only Lorentz transformations that leave a particle with k = 0 at rest.

- The second possibility is p^2 = 0.

One chooses k^μ = (1, 0, 0, 1), and the little group is ISO(2), the invariance group of Euclidean geometry (rotations and translations in two dimensions).
Any p^μ within a given class can be obtained from the corresponding k^μ by a Lorentz transformation.

Normalization: $$\langle \psi_{p',\sigma'}, \psi_{p,\sigma} \rangle = \delta_{\sigma\sigma'} \delta(p - p')$$

For m = 0, the little group is ISO(2). This is the case of interest here. It turns out that the quantum-mechanical states are only distinguished by the eigenvalue of J_3, the component of the angular momentum in the direction of motion

$$J_3 \psi_{k,\sigma} = \sigma \psi_{k,\sigma}$$

The eigenvalue σ is called the helicity

$$U(\Lambda, 0)\psi_{p,\sigma} = N e^{i\sigma\theta(\Lambda,p)} \psi_{\Lambda p,\sigma}$$

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Gravitons from representations of the Poincaré group

σ = ±1 characterizes the photon.

If a plane wave φ transforms as φ → $e^{-ih\theta}$ under a rotation around the direction of propagation, one calls h its helicity.
And because weak gravitational waves in a flat background transform like:

$$e'_R = e^{-2i\theta} e_R, e'_L = e^{2i\theta} e_L$$

we associate the particle with σ = ±2 with the gravitational interaction and call it the **graviton.**

Since for a massless particle |σ| is called its spin, we recognize that the graviton has spin 2

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Quantization of the linear field theory

We can start with superposition of plane-wave solutions and formally turn it into an operator

$$f_{\mu\nu} = e_{\mu\nu}e^{ikx} + e^*_{\mu\nu}e^{-ikx}$$

$$f_{\mu\nu}(x) = \sum_{\sigma=\pm 2} \sqrt{\frac{d^3k}{2|k|}} \left[a(k,\sigma)e_{\mu\nu}(k,\sigma)e^{ikx} + a^\dagger(k,\sigma)e^*_{\mu\nu}(k,\sigma)e^{-ikx}\right]$$

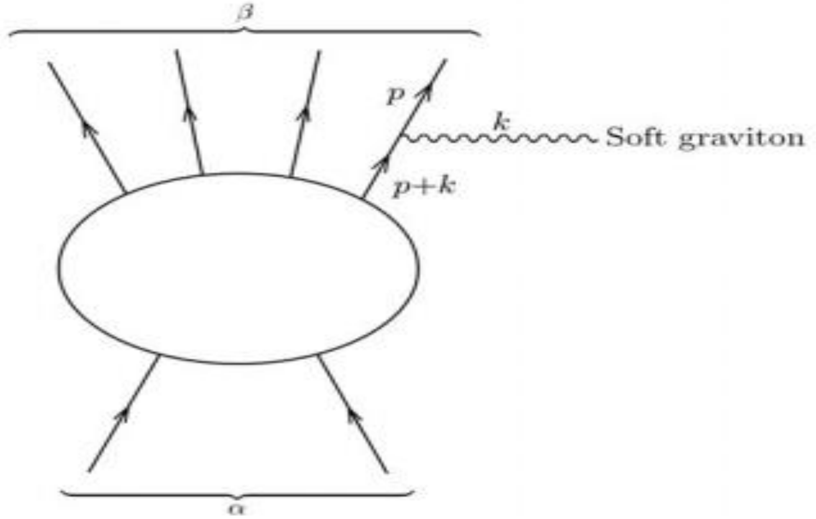
Where because of helicities ±2, $f_{\mu\nu}$cannot be a true tensor with respect to Lorentz transformations

$$f_{\mu\nu} \to \Lambda^\alpha_\mu \Lambda^\beta_\nu f_{\alpha\beta} - \partial_\mu c_\nu - \partial_\nu c_\mu,$$

In his paper Weinberg (1965) concluded that one can derive the equivalence principle (and thus GR if no other fields are present) from the Lorentz invariance of the spin-2 theory. No gauge invariance arguments are needed.
The gravitational mass mg is defined in this approach by the strength of interaction with a soft graviton hat is, a graviton with four-momentum k → 0.

Amplitude for the emission of a single soft graviton

$$M^{HV}_{\beta\alpha}(k) = M_{\beta\alpha} \sum_n \frac{\eta_n g_n p^\mu_n p^\nu_n}{p^2_n k_\mu - i\eta_n \epsilon}$$



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Quantization of the linear field theory

where $M_{\beta\alpha}$ denotes the amplitude for the process without soft-graviton emission; α refers to the ingoing and β to the outgoing particles ; g_n denotes the coupling of the graviton to particle n and p_n is the four-momentum of the nth particle.

From the Lorentz-invariance of the amplitude we can find that

$$\sum_n \eta_n g_n p_n^\nu = 0$$

Consequently, the couplings g_n must all be equal, and one can set $g_n \equiv \sqrt{8\pi G}$ Therefore, all low-energy particles with spin 2 and m= 0 couple to all forms of energy in an equal way.

Weinberg (1964) also showed that the effective gravitational mass m_g is given by:

$$m_g = 2E - \frac{m_i^2}{E}$$

where, as in Section m_i denotes the inertial mass, and E is the energy.

- For E → m_i, this leads to the usual equivalence of inertial and gravitational mass.
- At the same time, one has m_g = 2E for m_i → 0.

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Quantization of the linear field theory

Following Weinberg (1972), one can calculate the transition rate from the 3d level to the 1s level in the hydrogen atom due to the emission of a graviton. One needs at least the 3d level, since Δl = 2 is needed for the emission of a spin-2 particle.

$$\Gamma_g = \frac{P}{\hbar\omega}$$

One starts from the classical formula for gravitational radiation and interprets it as the emission rate of gravitons with energy $\hbar\omega$.

After the calculations we get:

$$\Gamma_g = \frac{Gm^3ec\alpha^6}{360\hbar^2} \approx 5.7 \times 10^{-40}\,\mathrm{s}^{-1}$$

Which corresponds to a lifetime of

$$\tau_g \approx 5.6 \times 10^{31}\ \mathrm{years}$$

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Path-integral quantization

The quantum-gravitational path integral, first formulated by Misner (1957), would be of the form

$$Z[g] = \int Dg_{\mu\nu}(x) e^{iS[g_{\mu\nu}(x)]}$$

Performing the Wick rotation one finds the following expression for the Euclidean gravitational action:

$$S_E[g] = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}(R-2\Lambda) = -\frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}K,$$

this action is unbounded, consider a conformal transformation of the metric, $g_{\mu\nu} \to \widetilde{g_{\mu\nu}} = \Omega^2 g_{\mu\nu}$. This yields:

$$S_E[g] = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}(\Omega^2 R + 6\Omega_{,\mu}\Omega^{,\mu} - 2\Lambda^4) - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}\Omega^2 K,$$

we recognize that the action can be made arbitrarily negative by taking into account large gradients of the conformal factor Ω. This is known as the conformal-factor problem.

Euclidean path integrals are often used in quantum cosmology, being related to boundary conditions of the universe

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The perturbative non-renormalizability

A major obstacle to the viability of perturbation theory is the non-renormalizability of quantum GR.

It turns out that the mass dimensionality (in units for which $\hbar$ = c = 1) of the coupling constant for a given interaction determines its renormalizability. This dimensionality is given by a coefficient Δ, which is called the **superficial degree of divergence** and which must not be negative. It can be calculated from the formula:

$$\Delta := 4 - d - \sum_f n_f(s_f + 1)$$

where d is the number of derivatives; n_f is the number of fields of type f; s_f = 0, 1/2, 1, 0 for scalars, fermions, massive vector fields, and photons and gravitons

For example, calculating Δ for standard QED interaction $-ie\overline{\psi}A_\mu\gamma^\mu\psi$ gives:

$$4 - 0 - \frac{3}{2} - \frac{3}{2} - 1 = 0$$

A theory is said to be renormalizable if these divergences can all be removed by a redefinition of a finite number of physical constants (masses, charges, etc.)
A non-renormalizable theory thus needs an infinite number of parameters to be determined experimentally.

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The perturbative non-renormalizability

In the background-field method to quantize gravity (DeWitt 1967), one expands the metric about an arbitrary curved background solution to the Einstein equations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{32\pi G} f_{\mu\nu}$$

If one chooses a flat background space–time $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$

one finds from the Einstein–Hilbert Lagrangian the Fierz–Pauli Lagrangian plus higher-order terms

$$\sqrt{32\pi G} f(\partial f)(\partial f) + \cdots + (\sqrt{32\pi G} f)^r (\partial f)(\partial f) + \cdots$$

These are infinitely many terms because the inverse of the metric, $g^{\mu\nu}$, enters the Einstein–Hilbert Lagrangian $\propto \sqrt{-g} R_{\mu\nu} g^{\mu\nu}$

Let's Considering the first interaction term $\sqrt{G} f(\partial f)(\partial f)$

We can find that is has negative mass dimension $\Delta = 4 - 2 - 3(0+1) = -1$

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Effective action and Feynman rules

If we apply the general path integral to the expansion $g_{\mu\nu} = \overline{g}_{\mu\nu} + \sqrt{32\pi G} f_{\mu\nu}$ we find an integral over the quantum field $f_{\mu\nu}$

After long calculations involving Grassmann path integral we get:

$$Z \propto \int \mathcal{D}f \, \Delta G[f] \exp\left(iS[f] - i\frac{1}{4\xi} \int d^4x \, G^\alpha G_\alpha \right)$$

the final path integral can be written in the form:

$$Z = \int \mathcal{D}f \, \mathcal{D}\eta_\alpha \, \mathcal{D}\eta_\alpha^* \, e^{iS_{\mathrm{tot}}[f,\eta,\overline{g}]}$$

Where:

$$\begin{aligned} S_{\mathrm{tot}}[f,\eta,\overline{g}] &= S[f,\overline{g}] - \frac{1}{4\xi} \int d^4x \, G^\alpha[f,\overline{g}] G_\alpha[f,\overline{g}] \\ &\quad + \int d^4x \, \eta_\alpha^*(x) A^{\alpha\beta}[f,\overline{g}](x) \eta_\beta(x) \\ &\equiv \int d^4x \, (L_g + L_{gf} + L_{\mathrm{ghost}}) \end{aligned}$$

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Effective action and Feynman rules

$$L_g = \sqrt{-g}\frac{R}{16\pi G} = \sqrt{-\overline{g}}\frac{\overline{R}}{16\pi G} + L_g^{(1)} + L_g^{(2)} + \cdots$$

('t Hooft and Veltman 1974)

$$L_g^{(1)} = \frac{f_{\mu\nu}}{\sqrt{32\pi G}}(\overline{g}^{\mu\nu}\overline{R} - 2\overline{R}^{\mu\nu})$$

This vanishes if the background is a solution of the (vacuum) Einstein equations

$$\begin{aligned} L_g^{(2)} =& \frac{1}{2}f_{\mu\nu;\alpha}f^{\mu\nu;\alpha} - \frac{1}{2}f_{;\alpha}f^{;\alpha} + f^{;\alpha}f_{\alpha\beta;\beta} \\ & - f^{\mu\beta;\alpha}f_{\mu\alpha;\beta} + \overline{R}\left(\frac{1}{2}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}f^2\right) + \overline{R}_{\mu\nu}\left(f f^{\mu\nu} - 2f_\mu^\alpha f_{\nu\alpha}\right) . \end{aligned}$$

The first line corresponds to the Fierz–Pauli Lagrangian while the second line describes the interaction with the background

$$L_{\mathrm{gf}} = \sqrt{-\overline{g}}\left(f_{\mu\nu;\nu} - \frac{1}{2}f_{;\mu}\right)\left(f^{\mu\rho}{}_{;\rho} - \frac{1}{2}f^{;\mu}\right) .$$

Gauge-fixing part

$$L_{\mathrm{ghost}} = \sqrt{-\overline{g}}\eta_\mu^*\left(\eta^\sigma{}_{\mu;\sigma;} - \overline{R}_{\mu\nu}\eta^\nu\right) .$$

The ghost part which is needed to guarantee the unitarity of the S-matrix

$$S_{\mathrm{tot}} = \int d^4x\sqrt{-\overline{g}}\left(\frac{\overline{R}}{16\pi G} - \frac{1}{2}f_{\mu\nu}D^{\mu\nu}_{\alpha\beta}f^{\alpha\beta} + \frac{f_{\mu\nu}}{\sqrt{32\pi G}}\int\sqrt{-\overline{g}}\,\overline{g}^{\mu\nu}\overline{R} - 2\overline{R}_{\mu\nu} + \eta_\mu^*(\overline{g}^{\mu\nu}2 - \overline{R}^{\mu\nu})\eta_\nu + O(f^3)\right) ,$$

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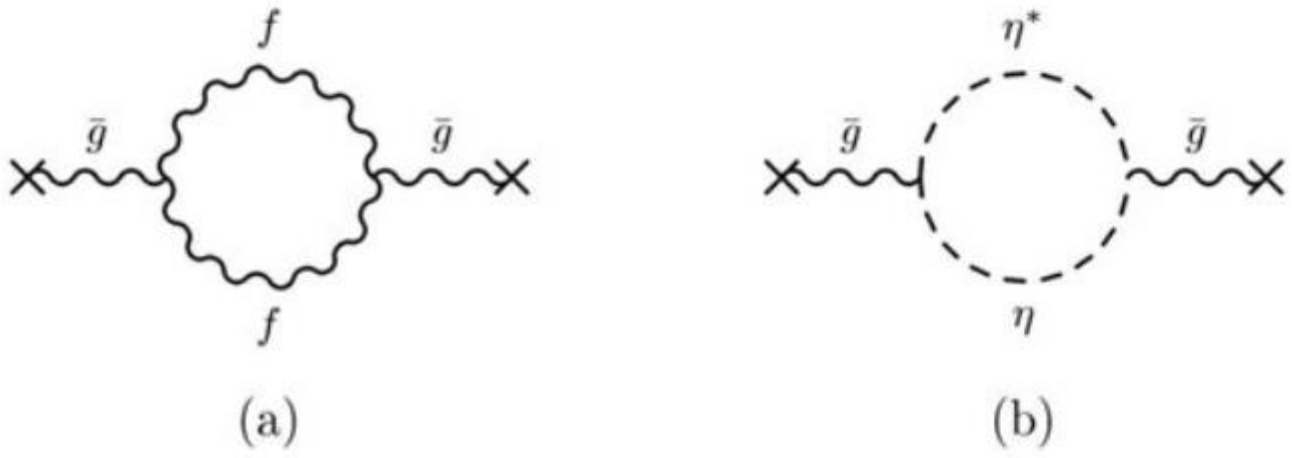
Effective action and Feynman rules

The action leads to diagrams with at most one loop such as those depicted (the 'one-loop approximation')
a) describes a graviton loop interacting with the background field.
b) describes a ghost loop interacting with the background.



In the case of a flat background, for which, for example, the propagator (in the harmonic gauge) is given by the expression:

$$D^{\mu\nu}_{\alpha\beta} = \frac{1}{2(k^2 - i\epsilon)}\left(\eta^\mu_\alpha \eta^\nu_\beta + \eta^\mu_\beta \eta^\nu_\alpha - \frac{2}{n-2}\eta^{\mu\nu}\eta_{\alpha\beta}\right).$$

The pole for n = 2 arises because in two space–time dimensions, GR is a topological theory.

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quantum-gravitational correction term to the Newtonian potential

Genuine predictions can, be obtained from the action in the infrared limit.
The first example is a quantum-gravitational correction term to the Newtonian potential, which was derived from linear quantum gravity by Bronstein (1936).

$$V(r) = -\frac{Gm_1m_2}{r}$$

After calculating the first correction term we get:

$$V(r) = -\frac{Gm_1m_2}{r}\left(1 + \frac{3G(m_1+m_2)}{rc^2} + \frac{4}{10\pi G\hbar}\frac{1}{r^2c^3} + O(G^2)\right)$$

Although arising from a one-loop amplitude, the first correction term is in fact an effect of classical GR.
It can be obtained from the Einstein–Infeld–Hoffmann equations, describing the approximate dynamics of a system of point-like masses due to their mutual gravitational interactions.

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Semiclassical Einstein equations

For a general quantum field φ (with possible components φ_i), the generating functional W[J] is defined by the path integral

$$\langle out, 0 | in, 0 \rangle J =: Z[J] =: e^{iW[J]} = \int \mathcal{D}\phi \, e^{iS[\phi] + iJ_k \phi^k}$$

where J is an external current

If we introduce effective action: $$\Gamma[\langle \phi \rangle] = W[J] - \int d^4x \, J(x) \langle \phi(x) \rangle$$

we can eventually get the effective field equation up to one-loop order:

$$\frac{\delta S}{\delta \langle \phi(x) \rangle} - i\hbar \frac{1}{2} G_{mn} S^{nmk} = -J(x)$$

$$\Big\downarrow$$

$$\frac{\delta S_{\mathrm{EH}}}{\delta \langle g_{\mu\nu}(x) \rangle} + \frac{\delta S_m}{\delta \langle g_{\mu\nu}(x) \rangle} = \frac{\sqrt{-g}}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{\sqrt{-g}}{2} T_{\mu\nu} \right)$$

In the case J = 0 one gets from the following semiclassical Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (\langle T_{\mu\nu} \rangle + \langle t_{\mu\nu} \rangle)$$

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Semiclassical Einstein equations

After the renormalization we get following semiclassical Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu} = 8\pi G(\langle T_{\mu\nu}\rangle_{\mathrm{ren}} + \langle t_{\mu\nu}\rangle_{\mathrm{ren}})$$

Where:

$$H^{(1)}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\int d^4x\,\sqrt{-g}\,R^2\right) = 2R_{\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}$$

$$H^{(2)}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\int d^4x\sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}\right) = 2R^{\alpha}_{\mu;\nu\alpha} - 2R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R + 2R^{\alpha}_{\mu}R_{\alpha\nu} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}$$

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Box 18.1 DERIVATIONS OF GENERAL RELATIVITY FROM GEOMETRIC VIEWPOINT AND FROM SPIN-TWO VIEWPOINT, COMPARED AND CONTRASTED

	Einstein derivation	*Spin-2 derivation*
Nature of primordial spacetime geometry?	Not primordial; geometry is a dynamic participant in physics	"God-given" flat Lorentz spacetime manifold
Topology (multiple connectedness) of spacetime?	Laws of physics are local; they do not specify the topology	Simply connected Euclidean topology
Vision of physics?	Dynamic geometry is the "master field" of physics	This field, that field, and the other field all execute their dynamics in a flat-spacetime manifold
Starting points for this derivation of general relativity	1. Equivalence principle (world lines of photons and test particles are geodesics of the spacetime geometry) 2. That tensorial conserved quantity which is derived from the curvature (Cartan's moment of rotation) is to be identified with the tensor of stress-momentum-energy (see Chapter 15).	1. Begin with field of spin two and zero rest mass in flat spacetime. 2. Stress-energy tensor built from this field serves as a source for this field.
Resulting equations	Einstein's field equations	Einstein's field equations
Resulting assessment of the spacetime geometry from which derivation started	Fundamental dynamic participant in physics	None. Resulting theory eradicates original flat geometry from all equations, showing it to be unobservable
View about the greatest single crisis of physics to emerge from these equations: complete gravitational collapse	Central to understanding the nature of matter and the universe	Unimportant or at most peripheral

Charles W. Misner, Kip S. Thorne, John Archibald Wheeler - Gravitation-Princeton University Press (2017)

მადლობა ყურადღებისათვის

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- PHYSICAL REVIEW - STEVEN WEINBERGERG - Photons and Gravitons in 8-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass