Anyons and Topological Quantum Field Theories

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Particle Quantum Statistics

What happens to the many particle wavefunction when particles are exchanged in a certain way?

- Fermions $\rightarrow$ Exchange of two fermions the wavefunction accumulates a minus sign
- Bosons $\rightarrow$ Exchange of two bosons the wavefuction remains the same

Usual argument goes like this: "If you exchange a pair of particles then exchange them again, you get back where you started. So the square of the exchange operator should be the identity, or one. There are two square roots of one: $+1$ and -1, so these are the only two possibilities for the exchange operator."

However this is considered to be incorrect. To understand the possibilities in exchange statistics we need to look at quantum physics from the Path integral point of view.

Particle Quantum Statistics, Path Integral

We define propagator:

$$\langle \boldsymbol{x}_f \mid U(t_f, t_i) \mid \boldsymbol{x}_i \rangle$$

- it must be unitary - meaning no amplitude is lost along the way
- it must obey composition

$$\langle \boldsymbol{x}_f \mid U(t_f, t_i) \mid \boldsymbol{x}_i \rangle = \int d\boldsymbol{x}_m \langle \boldsymbol{x}_f \mid U(t_f, t_m) \mid \boldsymbol{x}_m \rangle \langle \boldsymbol{x}_m \mid U(t_m, t_i) \mid \boldsymbol{x}_i \rangle$$

- then this propagator can be written as

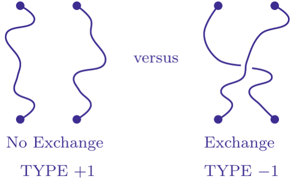
$$\langle \boldsymbol{x}_f \mid U(t_f, t_i) \mid \boldsymbol{x}_i \rangle = \mathcal{N} \int_{x_i}^{x_f} \mathcal{D}\boldsymbol{x}(t) e^{iS[\boldsymbol{x}(t)]/\hbar}$$

here $S[\boldsymbol{x}]$ is the classical action.

Particle Quantum Statistics, Two identical Particles

We know how to write path integral for one particle how can we generalize it to system with multiple identical particles?

- Denote state of two particle: $|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle$
- Suppose that path of particle look like this:



- We call the space of all states the configuration space $\mathcal{C}$. It divides up into topologically inequivalent pieces. Paths cannot be deformed into other paths by a series of small deformations.
- Paths can be composed with each other.

Particle Quantum Statistics, Two identical Particles

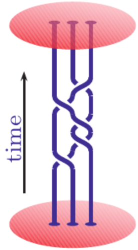
For propagators we can make two choices, that satisfy the unitarity and composability requirement:

- $\langle \boldsymbol{x}_{1f}, \boldsymbol{x}_{2f} \mid U \mid \boldsymbol{x}_{1i} \boldsymbol{x}_{2i} \rangle = \sum_{\text{type } +1 \text{ paths}} e^{\frac{i}{\hbar} S[\text{path}]} + \sum_{\text{type -1 paths}} e^{\frac{i}{\hbar} S[\text{path}]}$
- $\langle \boldsymbol{x}_{1f}, \boldsymbol{x}_{2f} \mid U \mid \boldsymbol{x}_{1i} \boldsymbol{x}_{2i} \rangle = \sum_{\text{type } +1 \text{ paths}} e^{\frac{i}{\hbar} S[\text{path}]} - \sum_{\text{type -1 paths}} e^{\frac{i}{\hbar} S[\text{path}]}$

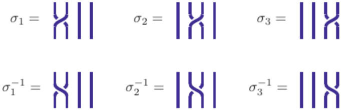
If we have + we say we have bosons, if we use - we say we have fermions. We can generalize this for many particles.

Particle Quantum Statistics, Many Identical Particles, Paths in $(2+1)$-Dimensions

Suppose as time goes by we shuffle identical particles and after some time they arrive at the same locations



- The path throught the configuration space of particles in two dimensions is knows as a braid.
- The set of braids have group structure.
- The generators of braids are:



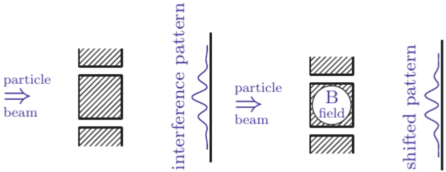
Particle Quantum Statistics, Braids and Permutations

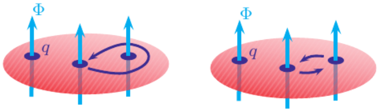
- In $(2+1)$-Dim. we have braids - B_N
- In $(3+1)$-Dim. we have permutation - S_N
- The Paths in Configuration space are divided by equivalence classes.
- For abelian case we introduce the scalar unitary repressentation ρ of G group then we have

$$\langle \{\boldsymbol{x}\}_f \,|\, U(t_f, t_i) \,|\, \{\boldsymbol{x}\}_i \rangle = \sum_{g \in G} \rho(g) \sum_{\text{paths} \in g} e^{\frac{i}{\hbar} S[\text{path}]}$$

- Mathematics cannot tell use what representation to use. Which representation actually occurs is a information about system being studied
- In $(2+1)$-Dim we have trivial($\rho = 1$), alternating $(\rho = \pm 1)$ and Anyons/fractional statistics $\rho = e^{i\theta W(g)}$
- In $(3+1)$-Dim we only have trivial and alternating meaning that we have only fermions and bosons.
- In Nonabelian case - we get degeneracies and $\rho(g)$ become matrices.

Aharonov-Bohm Effect and Charge-Flux Composites

- 

- 

- Particle aquires phase due to Vector Potential.

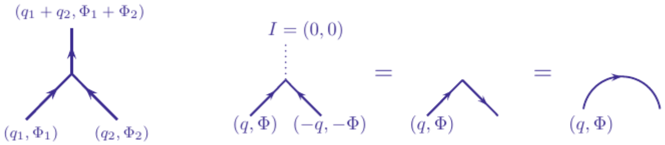
$$\exp\left[\frac{iq}{\hbar}\oint d\boldsymbol{l}\cdot\boldsymbol{A}\right] = \exp\left[\frac{iq}{\hbar}\Phi_{\text{enclosed}}\right]$$

- we have model abelian anyons as a charge-flux composites q, Φ

$$\text{Phase of encircling} = e^{2iq\Phi/\hbar}$$

$$\text{Phase of exchange} = e^{iq\Phi/\hbar} = e^{i\theta}$$

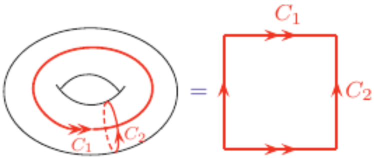
Fusion of Anyons, Anti-Anyons and the Vacuum Particle



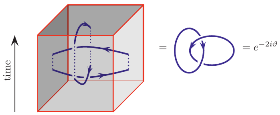
- Phase of dragging an anyon clockwise around an anyon is 2θ
- Phase of dragging an anti-anyon clockwise around an anti-anyon is 2θ
- Phase of dragging an anyon clockwise around an anti-anyon is -2θ

Anyon Vacuum on Torus and Quantum Memory

Suppose we have a torus



- Define T_1 operator - Creates particle-antiparticle pair and moves around C_1
- Define T_2 operator - Creates particle-antiparticle pair and moves around C_2



- $T_2 T_1 = e^{-2i\theta} T_1 T_2$
- Since those operators are unitary, on ground states: $T_1 |\alpha\rangle = e^{i\alpha} |\alpha\rangle$
- $T_2 |\alpha\rangle$ is also a ground state since $T_1 T_2 |\alpha\rangle = e^{i(\alpha+2\theta)} |\alpha + 2\theta\rangle$
- With this we generate ground states, they are degenerate. if $\theta = \pi \frac{p}{m}$ then there is m independent ground states.

Chern-Simons Theory Basics

We introduce a gauge field a_α known as the Chern-Simons vector potential.

- Lagrangian: $\mathcal{L} = \frac{k}{4\pi}\epsilon^{\alpha\beta\gamma}a_\alpha\partial_\beta a_\gamma - j^\alpha a_\alpha$
- Eq. Motion: $j^\alpha = \frac{k}{2\pi}\epsilon^{\alpha\beta\gamma}\partial_\beta a_\gamma$
- Chern-Simons magnetic field:
  $$b = \sum_{n=1}^{N}\frac{2\pi q_n}{k}\delta(\boldsymbol{x} - \boldsymbol{x}_n)$$
- We have generated Charge-Flux composite:
  $(q_n, \frac{2\pi q_n}{k})$

Lets write N particle path integral

- $\sum_{\text{paths}}\sum_{a_\mu} e^{i(S_0+S_{CS}+S_{\text{coupling}})/\hbar}$
- $\sum_{\text{paths}} e^{iS_0/\hbar + i\theta W(\text{paths})}$
- Thus "integrating out" the Chern-Simons gauge field implements fractional statistics by inserting a phase $e^{\pm i\theta}$ for each exchange.

Chern-Simons Theory Basics

The vacuum of an anyon theory knows about the statistics of the particles, even when particles are not present. That is the ground state degeneracy on the torus marches the number of particle species. Thus in the absece of particles, we will be interested in

$$Z(\mathcal{M}) = \int_{\mathcal{M}} \mathcal{D}a_\mu e^{\frac{i}{\hbar} S_{CS}}$$

This integral gives exactly the ground-state degeneracy of the system. It is a topological invariant of the spacetime manifold.

Nonabelian Chern-Simons Theory

We generalize abelian chern-simons theory by promoting the gauge field a_α to be a vector of matrices.

- $a_\mu(x) = a_\mu^a(x)\left(\frac{\sigma_a}{2i}\right)$
- $W_L = \mathrm{Tr}\left[P\exp\left(\oint dl^\mu a_\mu\right)\right]$
- $a_\mu \to U^{-1}a_\mu U + U^{-1}\partial_\mu U$

- This action in unique in being metric independent and (almost) gauge invariant

$$S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\epsilon^{\alpha\beta\gamma}\mathrm{Tr}\left[a_\alpha\partial_\beta a_\gamma + \frac{2}{3}a_\alpha a_\beta a_\gamma\right]$$

$$S_{CS} \to S_{CS} + 2\pi\nu k$$

$$\nu = \frac{1}{24\pi^2}\int_{\mathcal{M}} d^3x\epsilon^{\alpha\beta\gamma}\mathrm{Tr}\left[(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)(U^{-1}\partial_\gamma U)\right]$$

$$Z(\mathcal{M}) = \int_{\mathcal{M}} \mathcal{D}a_\mu(x)e^{iS_{CS}+2\pi i\nu k}$$

$$\text{Know Invariant } = \frac{Z(S^3), L_1, L_2}{Z(S^3)} = \frac{\int_{S^3}\mathcal{D}a_\mu W_{L_1}W_{L_2}e^{iS_{CS}}}{\int_{S^3}\mathcal{D}a_\mu e^{iS_{CS}}}$$

Defining Topological Quantum Field Theory

- We might say that

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \text{any correlator} \rangle = 0$$

- Another way to define $(2+1)$-dimensional TQFT is a set of rules that take input and give output that is invariant under smooth deformations. Bypassing Chern-simons field theory altogether and define TQFT simply as a mapping from a manifold with links to an output

$$\{\mathcal{M}, L_a, L_b\} \mapsto Z(\mathcal{M}, a, b)$$

- mathematicians define it as a functor from category of cordism to category of vector spaces

$$Z : \boldsymbol{Cob}(D+1) \to \boldsymbol{Vect}$$

TQFT Atiyah's Axioms

We consider spacetime manifolds with no particles in them. We have $D+1$ dimensional manifold $\mathcal{M}$ and D dimensional submanifold $\Sigma \subset \mathcal{M}$. We can think of Σ as being a slice at a fixed time.

Axiom 1: Associated to each D-dimensional manifold Σ is a Hilbert space $V(\Sigma)$ that does not change under continuous deformation of Σ.

Axiom 2: The Hilbert space associated with the disjoint union of two D-dimennsional manifold Σ_1 and Σ_2 is the tensor product of the Hilbert spaces associated with each manifold. That is

$$V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$$

Axiom 3: If $\mathcal{M}$ is a $D+1$-dimensional manifold with D dimensional boundary $\Sigma = \partial\mathcal{M}$, then we associate a particular element of the vector space $V(\Sigma)$ with this manifold. We werite

$$Z(\mathcal{M}) \in V(\partial\mathcal{M})$$

Axiom 4: Reversing orientation

$$V(\Sigma^*) = V^*(\Sigma)$$

TQFT

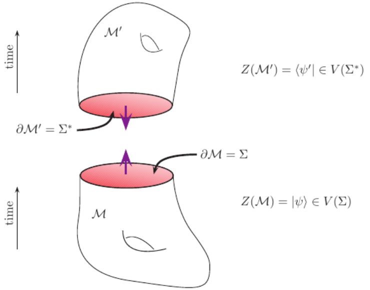
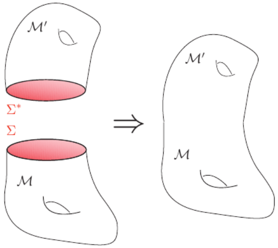


Fig. 7.3 In this picture $\mathcal{M}$ and $\mathcal{M}'$ are meant to fit together since they have a common boundary but with opposite orientation, $\Sigma = \partial\mathcal{M} = \partial\mathcal{M}'^*$. Here, $\langle\psi'| = Z(\mathcal{M}') \in V(\Sigma^*)$ lives in the dual space of $|\psi\rangle = Z(\mathcal{M}) \in V(\Sigma)$. Note that the normals are oppositely directed.



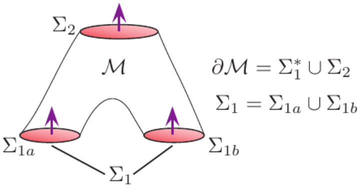
$$Z(\mathcal{M} \cup_\Sigma \mathcal{M}') = \langle Z(\mathcal{M}')|Z(\mathcal{M})\rangle = \langle\psi'|\psi\rangle$$

Fig. 7.4 Gluing two manifolds together by taking the inner product of the wavefunctions on their common, but oppositely oriented, boundaries.

TQFT Cobordism

Two Manifolds Σ_1, Σ_2 are called "cobordant" if after reversing the orientation of one of the manifolds, their disjoint union is the boundary of a manifold M

$$\partial \mathcal{M} = \Sigma_1^* \sqcup \Sigma_2$$



in this case we have

$$Z(\mathcal{M}) = \sum_{\alpha\beta} U^{\alpha\beta} \left|\psi_{\Sigma_2,\alpha}\right\rangle \otimes \left\langle\psi_{\Sigma_1,\beta}\right|$$

TQFT Cobordism

Let $\partial\mathcal{M} = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3$We can choose any subset of the disjoint boundaries as the "incoming" boundary set and the remaining boundaries as the "outgoing" boundaries

$\mathcal{M}$ is cob between $(\Sigma_1 \cup \Sigma_2)^*$ (incoming) and Σ_3 (outgoing)

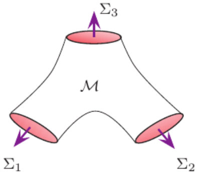
$$Z(\mathcal{M}) = \sum_{\alpha\beta\gamma} U^{\alpha\beta\gamma} \left|\psi_{\Sigma_3,\gamma}\right\rangle \otimes \left[\left\langle\psi_{\Sigma_1^*,\beta}\right| \otimes \left\langle\psi_{\Sigma_2^*,\alpha}\right|\right] \quad .$$

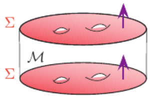
$\mathcal{M}$ is cob between Σ_1^* (incoming) and $\Sigma_2 \cup \Sigma_3$ (outgoing)

$$Z(\mathcal{M}) = \sum_{\alpha\beta\gamma} U^{\alpha\beta\gamma} \left[\left|\psi_{\Sigma_2,\alpha}\right\rangle \otimes \left|\psi_{\Sigma_3,\gamma}\right\rangle\right] \otimes \left\langle\psi_{\Sigma_1^*,\beta}\right| \quad .$$

$\mathcal{M}$ is cob between $\varnothing$ (incoming) and $(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$ (outgoir

$$Z(\mathcal{M}) = \sum_{\alpha\beta\gamma} U^{\alpha\beta\gamma} \left[\left|\psi_{\Sigma_1,\beta}\right\rangle \otimes \left|\psi_{\Sigma_2,\alpha}\right\rangle \otimes \left|\psi_{\Sigma_3,\gamma}\right\rangle\right]$$





$\mathcal{M} = \Sigma \times I$
$\partial\mathcal{M} = \Sigma^* \cup \Sigma$

$$Z(\Sigma \times I) = \text{Identity}$$

Bibliography

Topological Quantum - Steven H. Simon

Thank you for attention.