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Elementary Particle Physics II

INSTANTONS

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AGENDA

Introduction

Pontryagin index

The Baryon Asymmetry of the Universe

Summary

INTRODUCTION

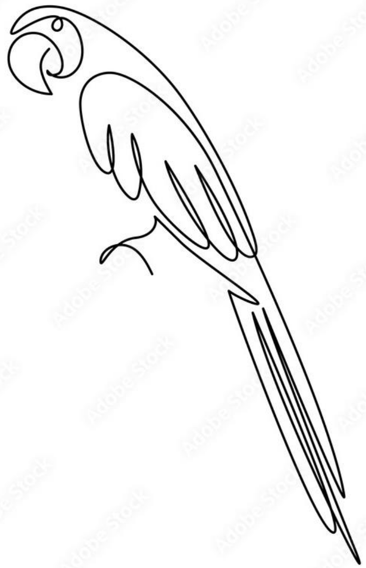
Main hero: SU(2)

One example of soliton solutions of gauge-field equations – localized in time as well as in space, which ‘t Hooft has christened ‘instantons’, ~~Polyakov(Russian theoretical physicist)~~ called it ‘pseudo particles’.

Let’s consider ‘Euclideanised’ space-time E^4 with boundary 3-sphere S^3.
Group space of $SU(2)$ is also S^3.
Hence topologically non-trivial solutions to the $\mathrm{SU}(2)$ gauge field equations are possible if there exist non-homotopic mappings of $S^3 \to S^3$, so

$$\pi_3(S^3) = \mathbb{Z}$$

Instantons are therefore possible in pure gauge theory, SSB is unnecessary.



- Euclidean space has coordinates (x_1, x_2, x_3, x_4), with $x_4 = -ix_0,\ \ x_0 = ct$
- Minkowski field tensor :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

- Dual of $F_{\mu\nu}$:

$$\widetilde{F_{\mu\nu}} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Using very left definitions under gauge transform

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}$$

$$F'_{\mu\nu} = SF_{\mu\nu}S^{-1}.$$

Defining the current:

$$K_\mu = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\left(A_\nu^a\,\partial_\rho A_\sigma^a + \frac{g}{3}\varepsilon_{abc}A_\nu^a\,A_\rho^b A_\sigma^c\right)$$

$$= \varepsilon_{\mu\nu\rho\sigma} Tr\left(\frac{1}{2}A_\nu\,\partial_\rho A_\sigma - \frac{ig}{3}A_\nu\,A_\rho\,A_\sigma\right)$$

Calculate $\partial_\mu K_\mu$.

$$\partial_\mu K_\mu = \varepsilon_{\mu\nu\rho\sigma} Tr\left(\frac{1}{2}(\partial_\mu A_\nu)(\partial_\rho A_\sigma) - ig\,(\partial_\mu A_\nu)A_\rho\,A_\sigma\right)$$

On the other hand,

$$\begin{aligned} \mathrm{Tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu} &= \tfrac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}\,\mathrm{Tr}\,\{\partial_{[\mu}A_{\nu]} - \mathrm{i}g[A_\mu, A_\nu]\}\{\partial_{[\kappa}A_{\lambda]} - \mathrm{i}g[A_\kappa, A_\nu]\} \\ &= 2\varepsilon_{\mu\nu\kappa\lambda}\,\mathrm{Tr}\,(\partial_\mu A_\nu)(\partial_\kappa A_\lambda) - 2\mathrm{i}g\varepsilon_{\mu\nu\kappa\lambda}\,\mathrm{Tr}\,A_\mu A_\nu(\partial_\kappa A_\lambda) \\ &\quad - 2\mathrm{i}g\varepsilon_{\mu\nu\kappa\lambda}\,\mathrm{Tr}\,(\partial_\mu A_\nu)A_\kappa A_\lambda - 2g^2\varepsilon_{\mu\nu\kappa\lambda}\,\mathrm{Tr}\,A_\mu A_\nu A_\kappa A_\lambda. \end{aligned}$$

Hence

$$\partial_\mu K_\mu = \frac{1}{4} Tr F_{\mu\nu}\widetilde{F_{\mu\nu}}$$

For a 4D volume V^4 in E^4, with boundary $\partial V^4 \sim S^3$

Pure vacuum $\rightarrow A_\mu = 0,\ F_{\mu\nu} = 0,$

then $K_\mu = 0.$

Now perform a gauge transformation again but at the boundary S^3

$$A_\mu \rightarrow -\frac{i}{g}(\partial_\mu S)S^{-1} \quad (on\ S^3)$$

boundary = ‘pure gauge’ ($F_{\mu\nu} = 0$). Then taking

$$S = \frac{x_4 + \mathrm{i}\mathbf{x}\cdot\boldsymbol{\sigma}}{\sqrt{\tau^2}}$$

where

$$\tau^2 = x_4^2 + \mathbf{x}^2.$$

We find:

$$\left.\begin{aligned} A_i &= \frac{\mathrm{i}}{g\tau^2}[x_i - \sigma_i(\boldsymbol{\sigma}\cdot\mathbf{x} + \mathrm{i}x_4], \\ A_4 &= -\frac{1}{g\tau^2}\boldsymbol{\sigma}\cdot\mathbf{x}, \end{aligned}\right\}$$

and

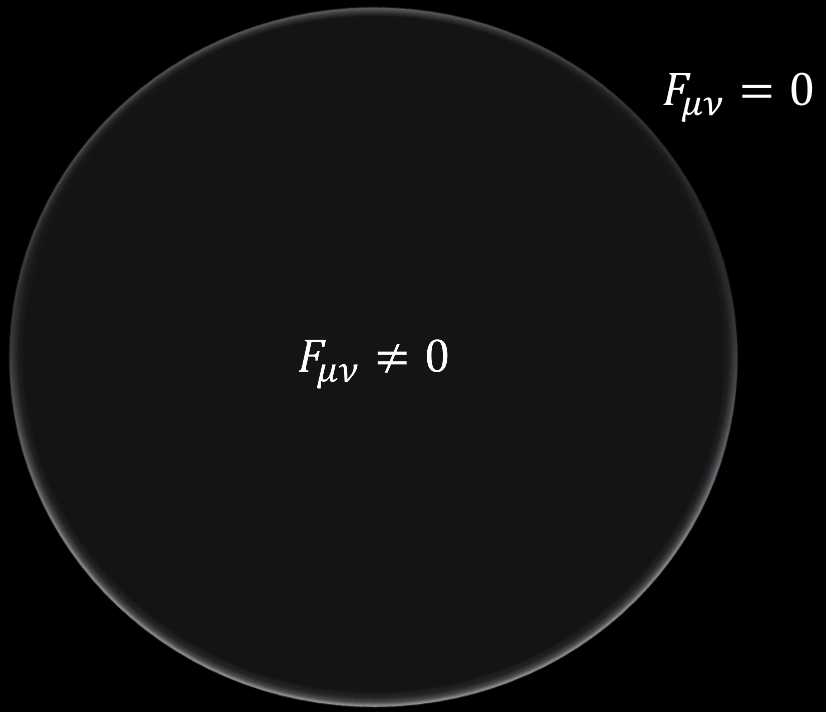
$$K_\mu = \frac{2x_\mu}{g^2\tau^4}.$$

If we integrate $\frac{1}{4}TrF_{\mu\nu}\widetilde{F_{\mu\nu}}$ on 4D volume:

$$\int_{V^4} TrF_{\mu\nu}\widetilde{F_{\mu\nu}}d^4x = 4\int_{V^4} \partial_\mu K_\mu\, d^4x$$

$$= 4\oint_{S^3} K_\mu d(area) = \frac{8\tau}{g^2\tau^4}2\pi^2\tau^3 = \boxed{\frac{16\pi^2}{g^2}}$$

Hence $F_{\mu\nu}$ cannot be zero over the whole volume, although it vanishes on the boundary.



The instanton. Field strength is non-vanishing inside the V^4, but vanishes on the boundary

The very last integral defines a topological index, aka Pontragin index:

$$q = \frac{g^2}{16\pi^2} Tr \int F_{\mu\nu} \widetilde{F_{\mu\nu}} d^4x$$

Considering we have:

$$q = \frac{g^2}{4\pi^2} \int \partial_\mu K_\mu d^4x = 1$$

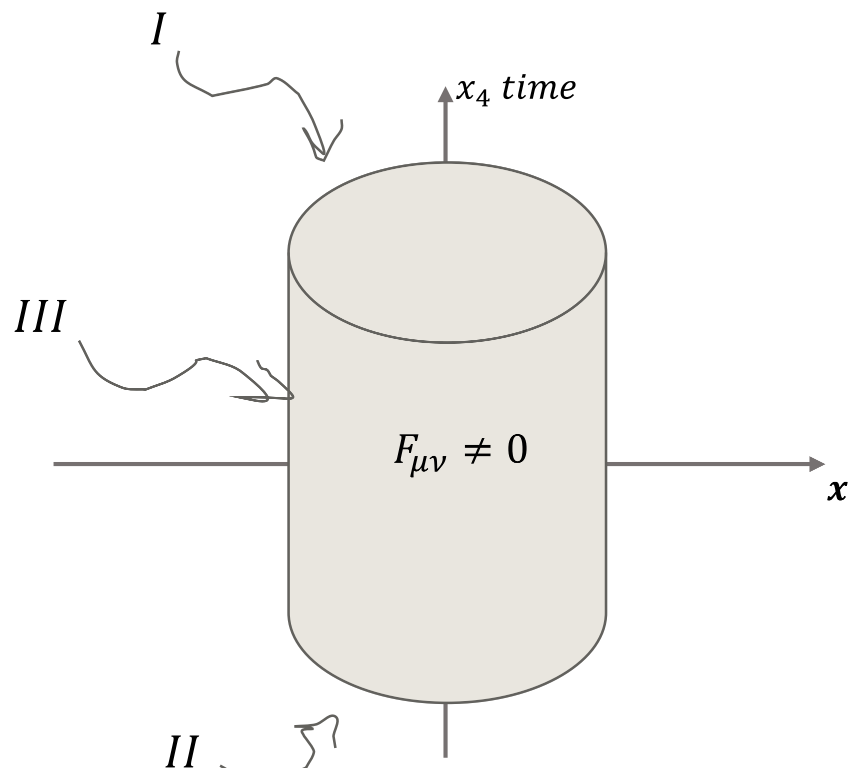
Put $A_\mu \to -\frac{i}{g}(\partial_\mu S)S^{-1}$ $(on\ S^3)$ in the definition of K_μ we get for q:

$$q = \frac{1}{24\pi^2} \int_G d^3g$$

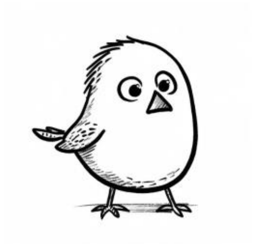
where d^3g is invariant group space volume element.
Hence q gives the degree of mapping $S^3 \to S^3$.
Let's interpret our solution as an evolution in time, rather than space.

E^4 has one time and three space dimensions. As one of these co-ordinates passes from $-\infty$ to $+\infty$, the field configuration changes.
If we interpret the present solution as an evolution in time, rather than space, we can redraw the S^3 boundary as given on the figure.

For q we are choosing a gauge in which the integral over the 'cylinder' III vanishes.
And after all the suffering in the end q reduces to the difference between two integrals, on the surfaces $x_4 \to \infty$ and $x_4 \to -\infty$.



The instanton boundary. I is $x_4 \to \infty$, II is $x_4 \to -\infty$.



The instanton describes a solution of the gauge-field equations in which, as x_4 evolves from $-\infty\ to + \infty$, a vacuum evolves into another vacuum (homotopy class changes from $(n-1)$ to n) and the Pontryagin index is:

$$q = n - (n-1) = 1$$

In the between region of these vacua field tensor $F_{\mu\nu}$ is non-vanishing, therefore there is positive field energy. Hence Yang-Mills vacua is infinitely degenerate and consists of an infinite number of homotopically non-equivalent vacua.

The instanton solution = transition from one vacuum class to another.

Special thanks to quantum tunneling for making the transition possible!

Tunneling amplitude for a single particle in 1D well in WKB approximation is:

$$e^{-\frac{1}{h}S_{euclidean}} = e^{\left\{-\frac{1}{h}\int_a^b [2m(V-E)]^{\frac{1}{2}}dx\right\}}$$

For $E > V,$ number of wave function oscillations is given by

$$\frac{1}{h}\int_a^b [2m(E-V)]^{\frac{1}{2}}dx = \frac{1}{h}\int_a^b pdx$$

On the other hand, $\int pdx = \int p\dot{x}dt = \int (H+L)\,dt =$/if total energy is normalized to zero/$= \int Ldt = S$

Difference between the last and the upper equations is the sign of $E - V$. Sign of V in eom $m\ddot{x} = -\frac{\partial V}{\partial x}$ is reversed by changing $t \rightarrow -t$.

Hence $S_{euclidean}$ is defined for imaginary times.

recalling
$$q = \frac{g^2}{16\pi^2}\mathrm{Tr}\int F_{\mu\nu}\tilde{F}_{\mu\nu}\,\mathrm{d}^4x.$$

So for the action of instanton:

Since $q = 1$

$$S = -\frac{1}{2}\int TrF_{\mu\nu}F_{\mu\nu}d^4x = -\frac{8\pi^2}{g^2}q = -\frac{8\pi^2}{g^2}$$

Hence the tunneling amplitude between the pure vacuum and the gauge rotated vacuum is of the order

$$\boxed{e^{-\frac{8\pi^2}{g^2}}}$$

So we have an infinite $\theta - vacua$ belonging to different homotopy classes. True ground state of Hillbert space may be written:

$$|vac\rangle_\theta = \sum_{n=-\infty}^{\infty} e^{i\pi\vartheta}\ |vac\rangle_n$$

where n is an integer labelling the homotopy class.

If $\vartheta \neq 0$ the vacuum states is complex, time reversal invariance is violated. From the CPT theorem, CP invariance is violated. Further, under P-transformation P is also violated.

't Hooft showed the axial current has an anomaly and the probability of baryon & lepton number violating decays is

$$e^{-\frac{16\pi^2}{g^2}} = 10^{-262}$$

REFERENCES

[1]. **Lewis H. Ryder**. *Quantum Field Theory*

[2]. **G. ‘t Hooft.** *Symmetry Breaking through Bell-Jackiw Anomalies* (Phys. Rev., 1976)

Thank you