Tbilisi State University

Fundamental Physics

Quantum Hall Skyrmions

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Non-linear Sigma Model

Consider the field $\boldsymbol{J} = (J_x, J_y, J_z)$ defined on a plane and subject to constraint

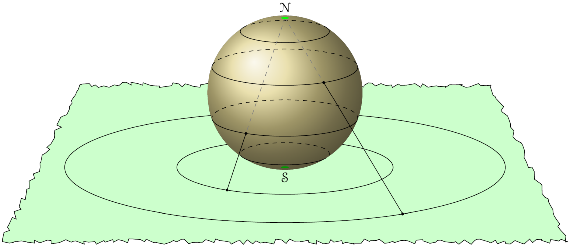
$$\boldsymbol{J}^2 = \sum_a J_a^2 = 1 \tag{1}$$

Non-linear sigma model is defined by the following energy functional

$$E = \frac{1}{2} \int (\partial_k J_a)^2 \, d\boldsymbol{r} \ . \tag{2}$$

In order the energy to be finite the derivatives $\partial_k J_a$ must vanish when $r \to \infty$. Thus the field $\boldsymbol{J}$ must approach a constant value as $r \to \infty$.

$O(3)$ SKYRMIONS



All point of the $r = \infty$ circle are mapped into single point $\boldsymbol{J} = (0, 0, 1)$. In such a cases we say that the original R^2 is compactified into S^2.

We have mapping $S^2 \to S^2$.

Such mappings are classified in accord with the homotopy group $\pi_2(S^2) = \mathbb{Z}$, where the integer counts the times the field space s^2 is wrapped when the coordinate r spans the entire plane.

Such configurations are called $O(3)$ skyrmions

Topological charge:

$$Q = \frac{1}{8\pi} \int \varepsilon_{abc} \varepsilon_{ij} J_a (\partial_i J_b)(\partial_j J_c) \, d\boldsymbol{r} \tag{3}$$

SKYRMIONS AND ANTISKYRMIONS

In order the energy to be minimal fields should satisfy equation:

$$\partial_i J_a \pm \varepsilon_{ij}\varepsilon_{abc} J_b \, \partial_j J_c = 0 \tag{4}$$

Solutions of this equation are

- skyrmions:

$$J_x = \frac{2\kappa^n r^n}{r^{2n}+\kappa^{2n}} \cos(n\phi) \ , \quad J_x = \frac{-2\kappa^n r^n}{r^{2n}+\kappa^{2n}} \sin(n\phi) \ , \quad J_z = \frac{r^{2n}-\kappa^{2n}}{r^{2n}+\kappa^{2n}} \tag{5}$$

  with the winding number $Q = n$;

- antiskyrmions:

$$J_x = \frac{2\kappa^n r^n}{r^{2n}+\kappa^{2n}} \cos(n\phi) \ , \quad J_x = \frac{2\kappa^n r^n}{r^{2n}+\kappa^{2n}} \sin(n\phi) \ , \quad J_z = \frac{r^{2n}-\kappa^{2n}}{r^{2n}+\kappa^{2n}} \tag{6}$$

  with the winding number $Q = -n$.

Grassmannian fields $G_{N,\nu}$

To define $G_{N,\nu}$ let's introduce ν amount of colums

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ \vdots \\ f_N^1 \end{pmatrix}, \quad f^2 = \begin{pmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ \vdots \\ f_N^2 \end{pmatrix}, \quad \cdots \quad f^\nu = \begin{pmatrix} f_1^\nu \\ f_2^\nu \\ f_3^\nu \\ \vdots \\ f_N^\nu \end{pmatrix}, \tag{7}$$

subject to

$$[f^r(\boldsymbol{r})]^\dagger f^s(\boldsymbol{r}) = \sum_\sigma^N [f_\sigma^r(\boldsymbol{r})]^* f_\sigma^s(\boldsymbol{r}) = \delta^{rs}. \tag{8}$$

Using these fields we can rewrite expression for the winding number as

$$Q = \frac{1}{2\pi i}\varepsilon_{kl} \sum_{\mu=1}^{N} \sum_{s=1}^{\nu} \int (\partial_k f_\mu^s)^* (\partial_l f_\mu^s) d\boldsymbol{r} \tag{9}$$

Non-commutative Grassmannian fields

Again, we introduce

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ \vdots \\ f_N^1 \end{pmatrix}, \quad f^2 = \begin{pmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ \vdots \\ f_N^2 \end{pmatrix}, \quad \cdots \quad f^\nu = \begin{pmatrix} f_1^\nu \\ f_2^\nu \\ f_3^\nu \\ \vdots \\ f_N^\nu \end{pmatrix}, \tag{10}$$

but in non-commutative case they are subject to

$$[f^r(\boldsymbol{r})]^\dagger \star f^s(\boldsymbol{r}) = \sum_{\sigma=1}^{N} [f_\sigma^r(\boldsymbol{r})]^* \star f_\sigma^s(\boldsymbol{r}) = \delta^{rs}. \tag{11}$$

where $\star$ product is defined as

$$g_1(\boldsymbol{r}) \star g_2(\boldsymbol{r}) = g_1(\boldsymbol{r}) e^{-\frac{i}{2}\theta\varepsilon_{nj} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g_2(\boldsymbol{r}) \tag{12}$$

Non-commutative winding number

Now we introduce so-called Moyal braket:

$$[g_1, g_2]_\star = g_1 \star g_2 - g_2 \star g_1 \ . \tag{13}$$

We can write

$$[f_\mu^{s\,\dagger}(\boldsymbol{r}), f_\mu^s(\boldsymbol{r})]_\star = -i\theta\varepsilon_{ij}(\partial_i f_\mu^{s\,\dagger}(\boldsymbol{r}))(\partial_j f_\mu^s(\boldsymbol{r})) + \mathcal{O}(\theta^2) \tag{14}$$

This suggests natural non-commutative generalization of Topological charge expression:

$$Q = \frac{1}{2\pi\theta}\sum_{\mu=1}^{N}\sum_{s=1}^{\nu}\int [f_\mu^{s\,\dagger}(\boldsymbol{r}), f_\mu^s(\boldsymbol{r})]_\star d\boldsymbol{r} \tag{15}$$

We can rewrite this as

$$Q(\boldsymbol{r}) = -\left(\hat{\rho}_{\mathrm{cl}}(\boldsymbol{r}) - \nu\rho_L\right), \tag{16}$$

where $\nu\rho_L$ represents the ground state value of the particle density and $\hat{\rho}_{\mathrm{cl}}(\boldsymbol{r}) - \nu\rho_L$ describes the excitation part of part of the charge density. Therefore, we have

$$Q = -\Delta N_e^{\mathrm{cl}} \tag{17}$$

where ΔN_e^{cl} is the number of extra particles brought by ecitations.

References

Z. F. Ezawa
"The Quantum Hall Effects, Field Theoretical Approach and Related Topics"

George Tsitsishvili.
Lecture notes: QH skyrmions