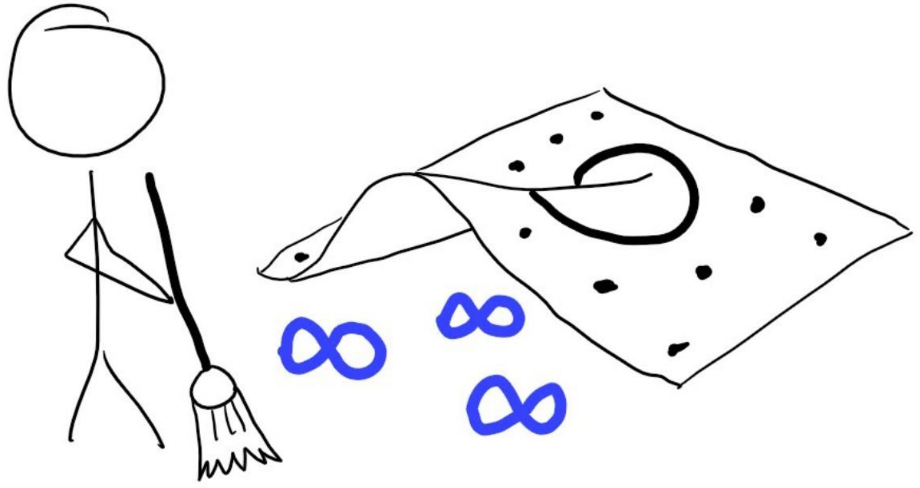
Divergences and Regularizations in QFT



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What causes the problem?

- While computing Feynman diagrams for different processes, we encounter a problem – Most loop amplitudes are infinite.
- QFT pioneers spent years struggling with infinities until the procedure of renormalization was developed.
- Regularization and renormalization procedures are used to extract meaningful physical results from calculations that seem to misbehave.

Regularization – introducing a high momentum cut-off to make integrals finite and regularize the theory.

Renormalization – absorbing extracted infinities into counterterms – re-scaling the theory to get meaningful results.

Literally cutting off ignorance

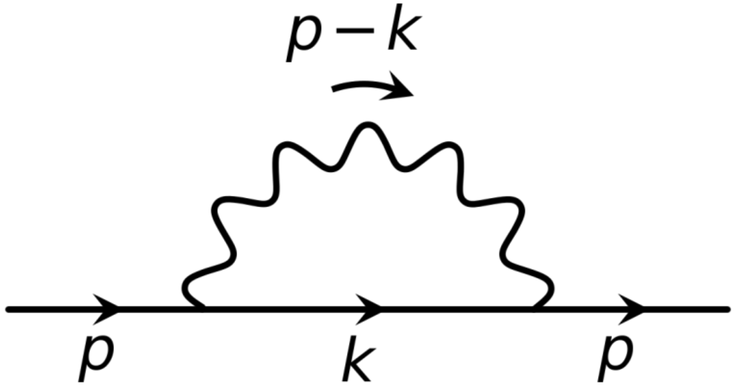
Example: the self-energy of the electron

From classical ED we know: $U = \frac{1}{2}e\varphi = \frac{e^2}{2R}$; $R = 0 \Rightarrow U \to \infty$

This is the limit for classical theory, it is time to take quantum effects into account.

QFT view:

$$iM = \int \frac{d^4k}{(2\pi)^4} \bar{u}(-ie\gamma^\mu) \frac{-ig_{\mu\nu}}{(p-k)^2} \frac{i(\gamma^\mu k_\mu - m)}{k^2 - m^2} (-ie\gamma^\nu) u$$



The one-loop correction to the electron propagator

- At large k, this goes $\int d^4k \frac{\gamma^\mu k_\mu}{k^4}$ which is linearly divergent.
- This type of divergence is called UV (ultra-violet) divergence and it corresponds to the infinities that arise when calculations involve integrating over large momentum values.

Closer look: Scattering matrix

Interaction picture:

Let's consider the theory of a real scalar field in n = 4 dimensions. It turns out that only interaction term which leads to a renormalizable theory must be quartic in the fields.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \lambda\varphi^4$$

$$S = T\,exp\left\{-i\int H_I\big(\varphi(x)\big)dx\right\} = Texp\left\{+i\int \mathcal{L}_I\big(\varphi(x)\big)dx\right\}$$

All information about the scattering (interaction) process is encoded in S-matrix. Its elements describe the transition amplitudes and their module square gives a measurable quantity – scattering cross section.

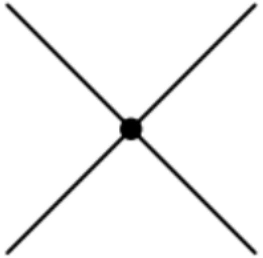
T – time ordering: $T\big(A(t_1), B(t_2)\big) = \begin{cases} A(t_1)B(t_2) & if \quad t_1 > t_2 \\ B(t_2)A(t_1) & if \quad t_2 > t_1 \end{cases}$

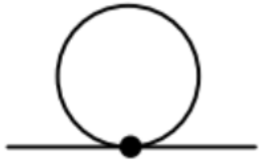
Expanding the S-matrix:

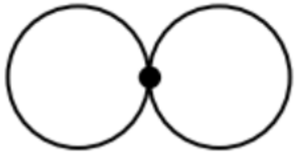
$$S = 1 + \int T\{i\lambda\, \varphi^4(x)\}dx + \frac{1}{2}\int T\{(i\lambda)^2\varphi^4(x)\varphi^4(y)\}dxdy + \ \ldots$$

1st order term according to Wick's theorem:

$$S_1(x) = i\lambda N\{\varphi^4(x)\} + i\lambda\overbrace{\varphi(x)\varphi(x)}N\{\varphi^2(x)\} + i\lambda(\overbrace{\varphi(x)\varphi(x)})^2$$



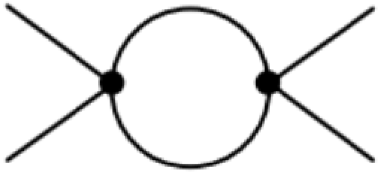




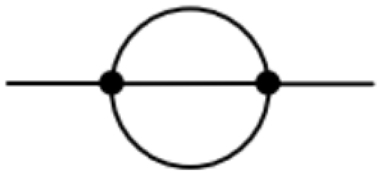
These are corresponding diagrams that we encounter in 1st order. Every closed loop corresponds to couplings.

2nd order diagrams $\sim\lambda^2$

No loops



1 loop



2 loops

Important remarks:

- Divergent integrals are always related to loop diagrams. Every loop corresponds to the integration in momentum space.
- Underlying reason: propagators are defined with chronological (T) ordering, but chronological ordering is not well defined when its time arguments coincide.

1st order divergent diagrams

$$\overbrace{\varphi(x)\varphi(x)} = -iD_c(0) = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon} \sim \int \frac{d^4k}{k^2} \sim k^2 \to \infty$$

generally, Feynman propagator - $\overbrace{\varphi(x)\varphi(0)} = -iD_c(x)$ contains different kinds of singularities, in particular, quadratic and logarithmic.

We solve this problem using Pauli-Villars Regularization

$$\frac{1}{k^2-m^2+i\varepsilon} \dashrightarrow \underbrace{\frac{1}{k^2-m^2+i\varepsilon} - \sum_n \frac{C_n}{k^2-M_n^2+i\varepsilon}}_{reg\Delta(k)}$$

Introducing new mass terms and constants will help us get rid of infinities later.

Regularizing $iD_c(0)$:

We are going to need two masses and two constants:

$$regD_c(0) = -\int\left(\frac{1}{k^2-m^2+i\varepsilon} - \frac{C_1}{k^2-M_1^2+i\varepsilon} - \frac{C_2}{k^2-M_2^2+i\varepsilon}\right)\frac{d^4k}{(2\pi)^4}$$

$$\sum_n C_n = 1\,; \quad \sum_n C_n M_n^2 = m^2$$

With these conditions we have:

$$C_1 = \frac{m^2-M_2^2}{M_1^2-M_2^2} \quad \text{and} \quad C_2 = \frac{M_2^2-m^2}{M_1^2-M_2^2}$$

Later on parametrization:

$$M_1^2 = M^2; \quad M_2^2 = \sigma\, M^2; \quad M \to \infty \qquad C_1 = \frac{-\sigma}{1-\sigma}\,; \quad C_2 = \frac{1}{1-\sigma}$$

For the explicit calculation of the integral we use:

$$\frac{1}{k^2-m^2+i\varepsilon} = \int_0^\infty e^{i(k^2-m^2+i\varepsilon)\alpha}\,d\alpha$$

Finally we get…

$$regD_c(0) = \frac{im^2}{16\pi^2}\left\{\frac{\sigma\ln(\sigma)}{1-\sigma}\,\frac{M^2}{m^2} + ln\frac{M^2}{m^2} + ln\sigma\right\}$$

From this we directly see quadratic and logarithmic divergences.

Finally, putting this into the matrix:

$$S = 1 + i\lambda\int N\{\varphi^4(x)\}\,dx + \frac{i\lambda m^2}{16\pi^2}\left\{\frac{\sigma\ln(\sigma)}{1-\sigma}\,\frac{M^2}{m^2} + ln\frac{M^2}{m^2} + ln\sigma\right\}\int N\{\varphi^2(x)\}\,dx + O(\lambda^2)$$

Divergent terms have the same operator structure as our initial, “unregularized” Lagrangian.

2nd order divergent terms:

$$S \sim \int S_2(x,y)dxdy$$

From second order would contribute:

$$\frac{9i}{4\pi}\lambda^2 ln\frac{M^2}{m^2}\int N\{\varphi^4(x)\}\,dx - \mathrm{i}\lambda^2 M^2\int N\{\varphi^2(x)\}dx +$$

$$+\mathrm{i}\lambda^2 ln\frac{M^2}{m^2}\int N\,(\partial_\mu\varphi)^2 dx$$

In higher terms the tendency is the same: structure of an initial lagrangian is kept, which leads us to re-scaling our theory and take regularized values also into account.

Our interaction term directly could have been:

$$\mathcal{L}_I{}' = \lambda\varphi^4 + \underbrace{\frac{9i}{4\pi}\lambda^2 ln\frac{M^2}{m^2}\varphi^4 - \mathrm{i}\lambda^2\varphi^2 + \lambda^2 ln\frac{M^2}{m^2}(\partial_\mu\varphi)^2}_{\text{Counter-terms}}$$

Re-scaling:

New, modified lagrangian:

$$\mathcal{L}' = \mathcal{L}'_0 + \mathcal{L}'_I = \frac{z_1}{2}(\partial_\mu \varphi)^2 - \frac{z_1 m'^2}{2}\varphi^2 - \lambda z_2 \varphi^4$$

Renormalization constants:

$$z_1 = 1 + 2\lambda^2 ln\frac{M^2}{m^2};$$

$$z_1 m'^2 = m^2 + 2\lambda^2 M^2$$

$$\lambda z_2 = \lambda + \frac{9}{4\pi}\lambda^2 ln\frac{M^2}{m^2}$$

Defining scalar field: $\varphi' = \sqrt{z_1}\varphi$

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \varphi')^2 - \frac{m'^2}{2}\varphi'^2 - \lambda' \varphi'^4$$

Where $\lambda' = \frac{z_2}{z_1}\lambda$

φ', m', λ' renormalized field, mass and coupling constant

References:

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- M.Srednicki; “Quantum Field Theory”
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- A.Zee; “Quantum Field Theory in a Nutshell”
- Quantum Field Theory Lectures of Sidney Coleman

THANK YOU