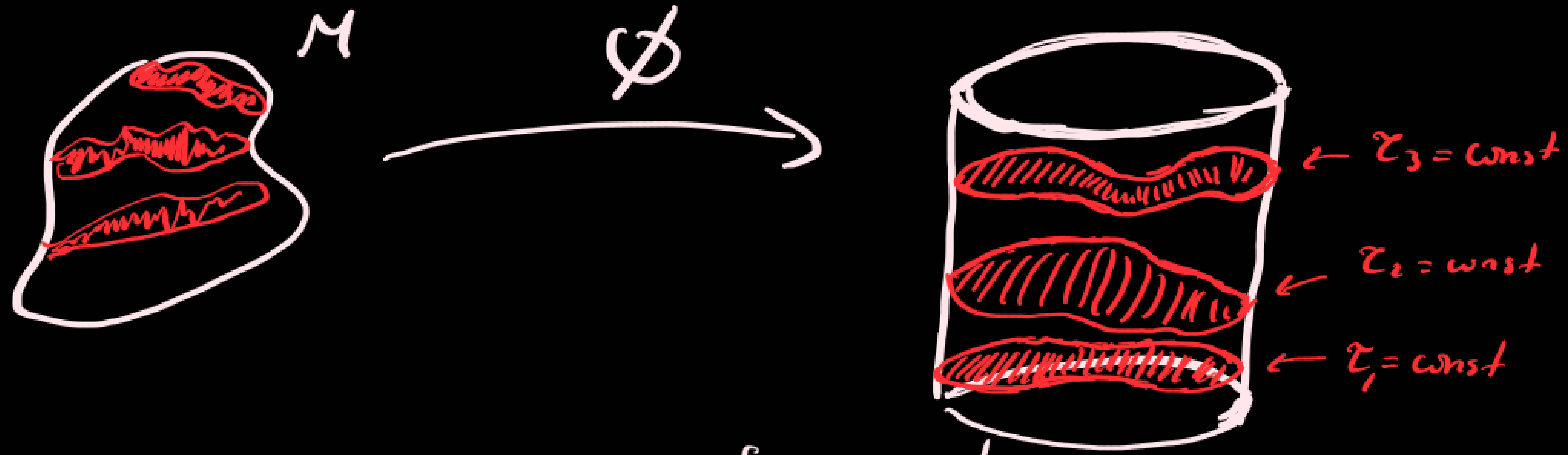
ADM Formalism of Gravity

(Hamiltonian Formalism)

FOLIATION OF 3-GEOMETRY

Our Universe (4D) $\qquad \mathbb{R} \times S$

time $\rightarrow \mathbb{R}$, $\quad S \leftarrow$ space

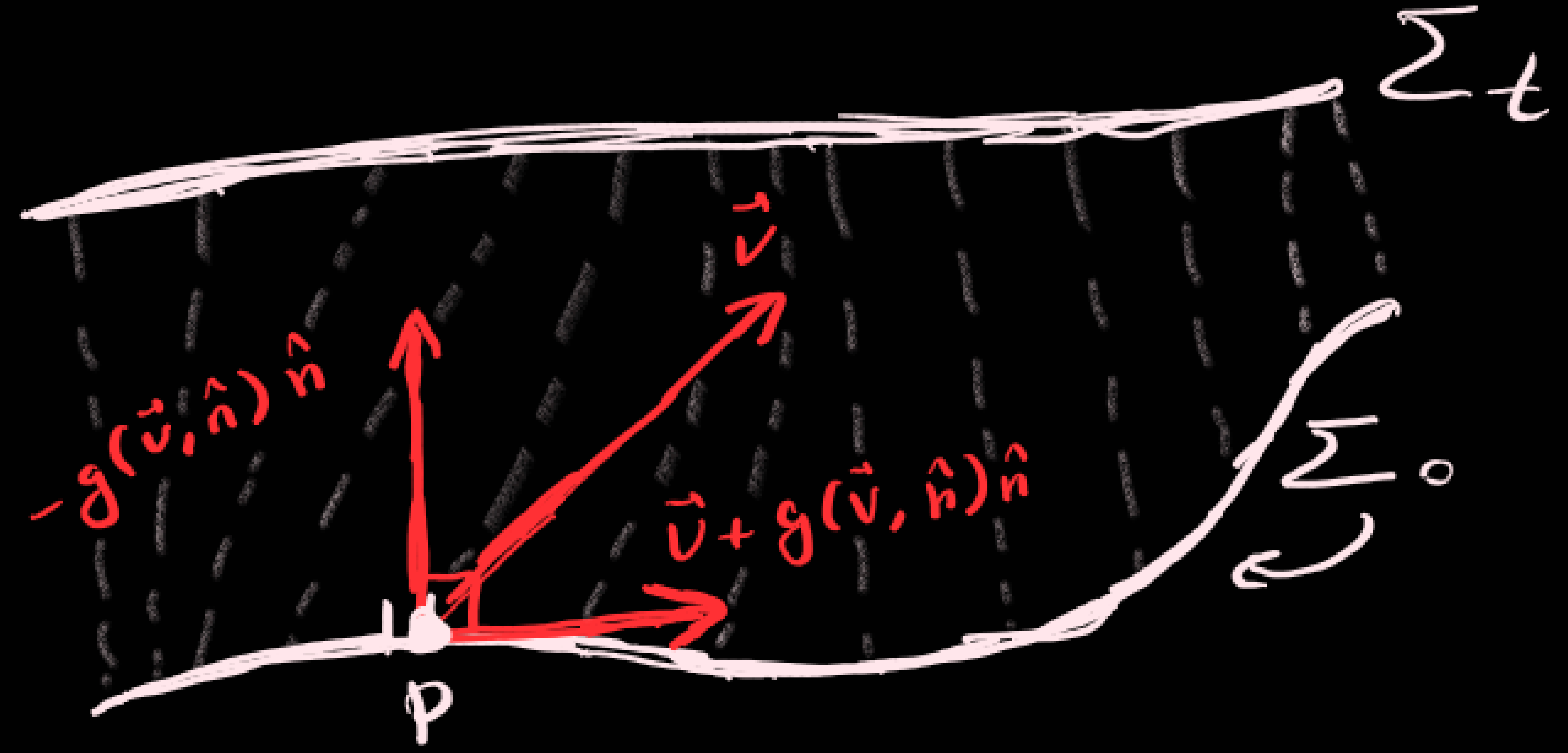


We call a $\Sigma \subset M$ a Slice if for some ϕ $\tau = t \circ \phi(p) = const$ for all $p \in \Sigma$

Idea: if We are given a Spacelike Slice of M at some particular time, can we evolve it in time? Answer is yes. But We need hamiltonian.

EXTRINSIC CURVATURE

$\Sigma_0 \subset M$ is spacelike meaning that $\forall s \in T\Sigma \quad g(s(p), s(p)) > 0$



$\hat{n}$ – tangent to Σ_0

$\vec{v} \in T_pM$ any vector

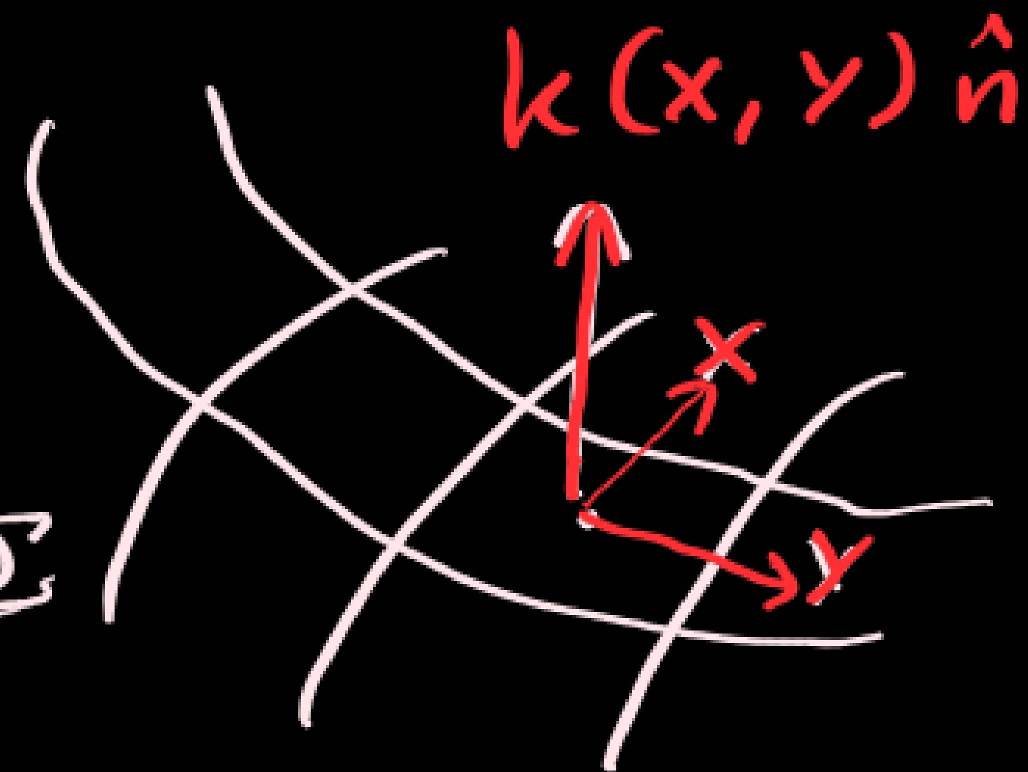
$$\vec{v} = -g(\vec{v},\hat{n})\,\hat{n} + (\vec{v} + g(\vec{v},\hat{n})\,\hat{n})$$

Similarly for any X, Y vector fields on Σ we have:

$$\nabla_X Y = \underbrace{-g(\nabla_X Y, n)}\,\hat{n} + (\nabla_X Y + g(\nabla_X Y, \hat{n})\,\hat{n})$$

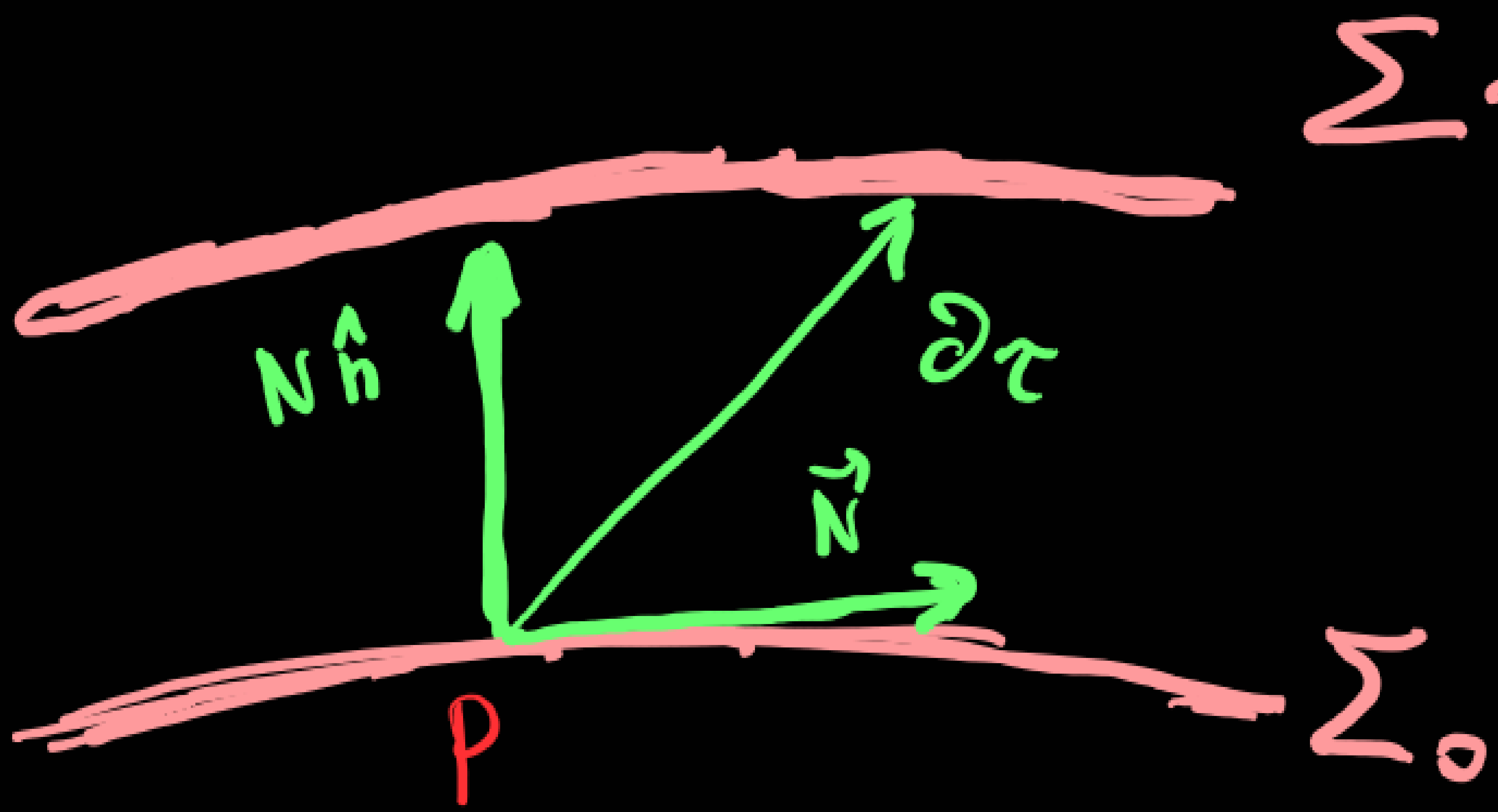
$$K(X,Y) = -g(\nabla_X Y, \hat{n})$$ ← Extrinsic Curvature of Σ

$$\overset{3}{\nabla}_X Y = \nabla_X Y + g(\nabla_X Y, \hat{n})\,\hat{n}$$ ← Covariant derivative on Σ



LAPSES, SHIFTS AND 3-METRIC

$M \xrightarrow{\phi} \mathbb{R} \times M$ define $\partial_\tau \equiv \phi_*^{-1} \partial_t$ to be a pull back of "time" derivative on $\mathbb{R} \times M$. then:



LAPSE: $-g(\partial_\tau, \hat{n}) = N$

SHIFT: $\vec{v} + g(\partial_\tau, \hat{n})\hat{n} = \vec{N}$

METRIC ON Σ: $g_{ij} = g(\partial_i, \partial_j)$

$\partial_1\ \partial_2\ \partial_3$ are tangent vectors on Σ

- We make a choice $(x^\mu) \longrightarrow$ $\partial_0 = \partial_\tau$; $\partial_i \partial_j \partial_k$ are tangent to Σ
- $(N, \vec{N})$ — Lapse and Shift
- 3-metric – $g_{ij} \equiv {}^3g$ We get Rieman tensor on $\Sigma \rightarrow {}^3R^i{}_{jmn}$
- Extrinsic Curvature – $k_{ij} = k(\partial_i, \partial_j)$

From these We obtain Gauss-Coduzzi equations:

$$R(\partial_i, \partial_j)\partial_k = ({}^3\nabla_i k_{jk} - {}^3\nabla_j k_{ik})\hat{n} + ({}^3R^m_{ijk} + k_{jk}k^m{}_i - k_{ik}k^m{}_j)\partial_m$$

And

$$k_{ij} = \frac{1}{2} N^{-1} (\dot{g}_{ij} - {}^3\nabla_i N_j - {}^3\nabla_j N_i)$$

CONSTRAINS AND DYNAMICAL EQUATIONS

From Branchi Identity: $\nabla^{\mu} G_{\mu\nu} = 0$

But this tells us:

$$\underbrace{\nabla^{0} G_{0\nu}}_{\text{at most } 1^{st} \text{ time derivative}} = - \underbrace{\nabla^{i} G_{i\nu}}_{\text{at most } 2^{nd} \text{ time derivative}}$$

But since Equations of Motion are second Order this means:

$$G_{0\mu} = 8\pi G\, T_{0\mu} \qquad \underbrace{G_{ij} = 8\pi G\, T_{ij}}$$

Constraint equation for Initial Data (q_{ij}, P_{ij})

Dynamical Equations that govern the evolution of (q_{ij}, P_{ij})

LAGRANGIAN AND HAMILTONIAN

$S = \int d^4x \underbrace{\sqrt{-g}\, R}$ (We consider Vacuum)

$\hookrightarrow \mathcal{L} = \sqrt{-g}\, R$ is lagrangian and then:

$$\mathcal{L} = (\mathrm{Det}\, g_{ij})^{1/2} N \left({}^3R + \mathrm{Tr}(K^2) - (\mathrm{Tr}(K))^2 \right)$$

Since K contains g_{ij} then: $p^{ij} = \dfrac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} = \sqrt{\mathrm{Det}\, g_{ij}} \left(K^{ij} - \mathrm{Tr}(K)\, g^{ij} \right)$

legendre transform:
$$H = (\mathrm{Det}\, g_{ij})^{1/2} (NC + N^i C_i)$$

$$C = -{}^3R + (\mathrm{Det}\, g_{ij})^{-1} \left(\mathrm{tr}(p^2) - \frac{1}{2} \mathrm{tr}(p)^2 \right) \quad C_i = -2\, {}^3\tilde{\nabla}^j \left((\mathrm{Det}\, g_{ij})^{-1/2} p_{ij} \right)$$

C and C_i are Hamiltonian and Diffeomorphic constrains.

Because in Vacuum $C, C_i = 0 \implies H = 0$

TIME EVOLUTION

Time evolution on Phase space (g_{ij}, p^{ij}) is defined by the Poisson Brackets: for continuous variable its:

$$\{f,g\} = \int_\Sigma \left\{ \frac{\partial f}{\partial p^{ij}(x)} \frac{\partial g}{\partial g_{ij}(x)} - \frac{\partial f}{\partial g_{ij}(x)} \frac{\partial g}{\partial p^{ij}(x)} \right\} \sqrt{\mathrm{Det}\, g_{ij}}\, d^3x$$

and then:

$$\{H, g_{ij}\} = \dot{g}_{ij} = 2(\mathrm{Det}\, g_{ij})^{-1/2} N \left(p_{ij} - \frac{1}{2} p^k_k g_{ij}\right) + 2\left({}^3\nabla_i N_j - {}^3\nabla_j N_i\right)$$

$$\begin{aligned}\{H, p^{ij}\} = \dot{p}^{ij} = & -N(\mathrm{Det}\, g_{ij})^{1/2}\left({}^3R^{ij} - \frac{1}{2}\, {}^3R\, g^{ij}\right) + \frac{1}{2} N (\mathrm{Det}\, g_{ij})^{-1/2} g^{ij}\left(p_{ab}p^{ab} - \frac{1}{2}(p^a_a)^2\right) \\ & - 2N(\mathrm{Det}\, g_{ij})^{-1/2}\left(p^{ia} p^j_a - \frac{1}{2} p^a_a p^{ij}\right) + (\mathrm{Det}\, g_{ij})^{1/2}\left(\nabla^i \nabla^j N - g^{ij}\, {}^3\nabla^a\, {}^3\nabla_a N\right) \\ & + (\mathrm{Det}\, g_{ij})^{1/2}\, \nabla_a\left([\mathrm{Det}\, g_{ij}]^{-1/2} N^a p^{ij}\right) - 2p^{ai}\, {}^3\nabla_a N^j + 2p^{aj}\, {}^3\nabla_a N^i\end{aligned}$$

Pheeeeww .....

Point is even if $H=0$, the time evolution is nontrivial

FIRST STEPS TO QUANTUM GRAVITY

$\{p^{ij}(x), q_{kl}(y)\} = (\delta^i_k \delta^j_l + \delta^i_l \delta^j_k)\,\delta^3(x-y)$ $\qquad$ $[\hat{p}^{ij}(x), \hat{q}_{kl}(y)] = -i\,(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k)\,\delta^3(x-y)$

$\{p^{\cdot\cdot}(\cdot), p^{\cdot\cdot}(\cdot)\} = 0$ $\qquad$ $[\hat{p}^{\cdot\cdot}(\cdot), \hat{p}^{\cdot\cdot}(\cdot)] = 0$

$\{g^{\cdot\cdot}(\cdot), g^{\cdot\cdot}(\cdot)\} = 0$ $\qquad$ $[\hat{g}^{\cdot\cdot}(\cdot), \hat{g}^{\cdot\cdot}(\cdot)] = 0$

$\{C(\vec{N}), C(\vec{N}')\} = C([\vec{N}, \vec{N}'])$ $\qquad$ $\{\hat{C}(\vec{N}), \hat{C}(\vec{N}')\} = -i\,\hat{C}([\vec{N}, \vec{N}'])$

$\{C(\vec{N}), C(N)\} = C(\vec{N} N)$ $\qquad$ $\{\hat{C}(\vec{N}), \hat{C}(N)\} = -i\,\hat{C}(\vec{N} N)$

$\{C(N), C(N')\} = C((N \partial^i N' - N' \partial^i N)\partial_i)$ $\qquad$ $\{\hat{C}(N), \hat{C}(N')\} = -i\,\hat{C}((N \partial^i N' - N' \partial^i N)\partial_i)$

$$C(N) = \int_\Sigma N C (\mathrm{Det}\, g_{ij})^{1/2} d^3x$$

where $\qquad \Rightarrow \quad \hat{H} = \hat{C}(N) + \hat{C}(\vec{N})$

$$C(\vec{N}) = \int_\Sigma N^i C_i (\mathrm{Det}\, g_{ij})^{1/2} d^3x$$

We say that a vector $\Psi \in L^2(\mathrm{Met}(\Sigma))$ is a physical state if

$$\boxed{\hat{H}\Psi = 0 \quad \text{for all } (N, \vec{N})} \longleftarrow \text{Wheeler-DeWitt equation}$$

APPLICATION OF ADM FORMALISM

* Quantum Gravity
* Numerical General Relativity
* Cosmology
* E.T.C.

References:

(1) Gauge Fields, Knots and Gravity – J. Baez.

Thank you