Relativistic Stars in General theory of Gravity

(Stellar equilibrium configurations, Newtonian and General relativistic formulations)

Elene Saralidze

Abstract

We study static, spherically symmetric equilibrium configurations in Newtonian, neo-Newtonian, and the full relativistic theory by solving the equilibrium equations for both three approaches and calculating the mass–radius diagrams for some simple neutron stars' equations of state.

Also we consider extended theories of gravity (ETG) and derive Tolman–Oppenheimer–Volkoff generalization equations for any ETG.

Introduction

Classical general relativity (GR) is a very elegant theory which is described by the Einstein field equations. They show the relation between the geometry of spacetime and the matter (fields) contribution. So far, we still do not have any clear evidence against GR though there are, of course, many alternative theories.

For static configurations GR is able to describe the structure of objects like stars. The GR solutions for isotropic stars are known as the Tolman–Oppenheimer–Volkoff (TOV) equations from which we can solve the equilibrium configuration of the stellar interior.

In general, nuclear reactions within the stellar interior induce energy flow via radiative convection. However, since for compact stars the nuclear timescale is much larger than the thermal and dynamical timescales one can assume the hydrostatic equilibrium for most of the star lifetime.

Newtonian and neo-Newtonian hydrodynamics

In a first approximation, hydrostatic equilibrium in stars can be studied with Newtonian mechanics. From this approach one obtains the Lane–Emden equation which is basically the Newtonian limit of the TOV system when pressure does not source gravitational effects in the stellar interior.

The basic equations of Newtonian hydrodynamics for an inviscid perfect fluid are the following:

$$\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0,$$

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \equiv \rho[\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v}] = -\nabla p,$$

Where ρ is the fluid density, p its pressure, and v field velocity.

Gravitational interaction is coupled to Euler's equation as:

$$\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla \Psi,$$

where the gravitational potential Ψ obeys the Poisson equation:

$$\nabla^2 \Psi = 4\pi G \rho.$$

The above equations system is suitable to study cosmology adopting the velocity field (Hubble Law) $\mathbf{v}=H(t)\mathbf{r}$,

$H(t) = \frac{\dot{a}(t)}{a(t)}$,

$a(t)$ is the scale factor,

Then we can write the Friedmann equations which reads:

$$\frac{\dot{a}^2}{a^2} + \frac{(-2E)}{a^2} = \frac{8\pi G}{3}\rho \quad \text{and} \quad \dot{H} + H^2 = -\frac{4\pi G}{3}\rho,$$

Where E is the constant of integration. the pressure is not dynamically relevant for the homogeneous and isotropic background.

Neo-Newtonian formalism relies on the following substitutions:

- Redefine the concept of internal and passive-gravitational mass density

$$\rho_i \rightarrow \rho + p,$$

- Redefine the active gravitational mass density, that sources the gravitational field

$$\rho_g \rightarrow \rho + 3p,$$

The final form for the fluid equations in this approach:

$$\dot{\rho} + \nabla \cdot \left(\rho \vec{v}\right) + p\nabla \cdot \vec{v} = 0,$$

$$\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla\Psi - \frac{\nabla p}{\rho + p},$$

$$\nabla^2 \Psi = 4\pi G\,(\rho + 3p).$$

Combining Eqs, we can obtain equations for the expansion of the homogeneous and isotropic background that are exactly the same as the relativistic Friedmann equations:

$$\frac{\dot{a}^2}{a^2} + \frac{(-2E)}{a^2} = \frac{8\pi G}{3}\rho,$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p).$$

Hydrostatic Equilibrium

In hydrostatic equilibrium thermal expansion energy is counteracted by the gravitational forces. Spherically symmetric, and static geometries are perfect to study astrophysical objects like stars. Lets review the Newtonian and general relativistic formulations.

Static gravitational equilibrium equations when general relativity is adopted are known as Tolman-Oppenheimer-Volkoff(TOV) equation

For a static symmetry perfect fluid the metric components are similar to those for the Schwarzschild metric:

$$c^2\,d\tau^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = e^\nu c^2\,dt^2 - e^\lambda\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2$$

By the perfect fluid assumption, the stress-energy tensor is diagonal with eigenvalues of energy density and pressure:

$$T_0^0 = \rho c^2$$

$$T_i^j = -P\delta_i^j$$

To process the further derivations, we solve Einstein's equations: ([2],[5])

$$\frac{8\pi G}{c^4} T_{\mu\nu} = G_{\mu\nu}$$

Assume perfect isotropic fluid

$$\nabla_\mu T^\mu_\nu = 0. \qquad \partial_\phi P = \partial_\theta P = 0$$

We obtain:

$$0 = \nabla_\mu T^\mu_1 = -\frac{dP}{dr} - \frac{1}{2}\left(P + \rho c^2\right)\frac{d\nu}{dr} \qquad \frac{d\nu}{dr} = \frac{1}{r}\left(1 - \frac{2Gm}{c^2 r}\right)^{-1}\left(\frac{2Gm}{c^2 r} + \frac{8\pi G}{c^4} r^2 P\right)$$

TOV equation:

$$\boxed{\frac{dP}{dr} = -\frac{G}{r^2}\left(\rho + \frac{P}{c^2}\right)\left(m + 4\pi r^3 \frac{P}{c^2}\right)\left(1 - \frac{2Gm}{c^2 r}\right)^{-1}}$$

The TOV equation is coupled to the mass definition:

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho,$$

M(R) is the total mass contained inside radius r, as measured by the gravitational field by a distant observer. It has the same form for Newtonian counterpart. If we take a limit for $c \rightarrow \infty$, we get Newtonian hydrostatic equilibrium equation

$$\frac{dp}{dr} = -\frac{G\rho M(r)}{r^2},$$

For the neo-Newtonian equilibrium we can derive the equations

$$\frac{dp}{dr} = -\frac{G(\rho + p)}{r^2}\tilde{M}(r),$$

Where

$$\frac{d\tilde{M}(r)}{dr} = 4\pi r^2(\rho + 3p).$$

Numerical Results *(Neutron Stars)*

Now our goal is to solve the differential equations for the internal structure of stars and compare the solution for the Newtonian formulation the full relativistic TOV equations. ([1])

White dwarfs are objects in which the simple Newtonian formalism works quite well. Although Neutron stars are perfect laboratories for testing the relativistic corrections incorporated by the TOV equations.

- After the specifying equation of state (EoS) $p(\varrho)$ for the stellar interior, the solution will also depends on specifying the central stellar pressure p(r=0)=p0, given the value P0 one can obtain total mass and radius of star. Unfortunately the inner composition of neutron stars are not well understood and the correct neutron star EoS is unknown.
- There are common ways to obtain EoS of nuclear matter based on relativistic mean-field theories and nucleon-nucleon potentials. One possibility is that precise observations of spin rate, mass and radius of many different stars can lead the reconstruction of the neutron star properties.

We consider a pure neutron configuration and use Fermi gas model as the Oppenheimer-Volkoff model. It corresponds to a polytrope:

$$p \sim \rho^{\gamma} \qquad \gamma = 5/3.$$

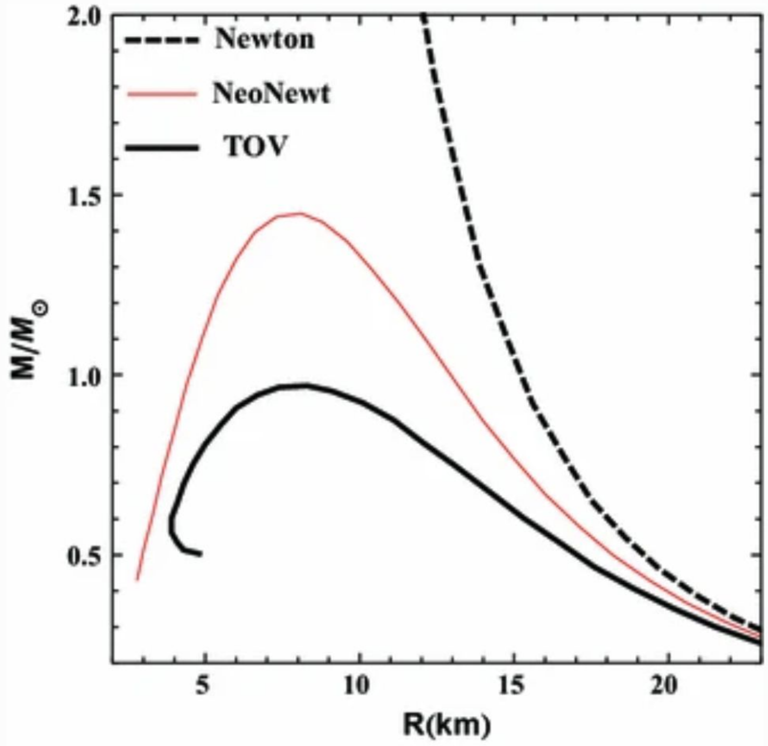
Write the dimensionless EoS equation under this consideration: ([4])

$$\frac{\bar{\rho}(\bar{p})}{c^2} = \bar{K}^{-1}\,\bar{p}^{3/5},$$

Where $\bar{K} = 1.914.$ $p = \epsilon_0 \bar{p}$ $\rho = \epsilon_0 \bar{\rho},$

$\epsilon_0 = 1.603 \times 10^{38}$ ergs/cm^3

For this EoS, plot the mass-radius diagram (fig.1)



The quantitative equivalence between the TOV and the neo-Newtonian models does not occur. More massive configurations are allowed in the neo-Newtonian context. However, a remarkable result is that the neo-Newtonian solution also presents a maximum mass which is a typical relativistic aspect. This shows that it can be used as a first approximation to the problem.

Here $M_{max} \sim 0.95 M_\odot$ at $R \sim 8$ km

Extended theories of gravity

- GR is the theory responsible for describing the gravitational interaction. However, it seems that building a successful model for the dynamics of the universe using GR and known matter fields as the source of Einstein equations is not enough to describe many issues that recently appeared in fundamental physics, astrophysics, and cosmology. ETG goal is encompass problems like inflation, dark energy, dark matter, quantum gravity…
- As we see there are problematic issues concerning astrophysical objects like neutron stars. The existence of neutron stars—without strong magnetic fields—with masses larger than 2M is challenged within the framework of GR.

We term extended theories of gravity (ETG) any alternative to GR in which the field equations can be recast in the form ([1],[4])

$$\sigma(\Psi^i)(G_{\mu\nu} - W_{\mu\nu}) = \kappa T_{\mu\nu},$$

- $\sigma(\Psi^i)$: Curvature invariant/other field
- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$
- $W_{\mu\nu}$: The symmetric tensor, stands for additional geometrical terms which may appear in the specific ETG under consideration.
- $\kappa = -8\pi G,$
- $T_{\mu\nu}$: Energy-momentum tensor

Note that this equation does not encompass all the possible alternatives to GR at the field equations level. As an example, for theories which have a time dependent effective gravitational coupling $\sigma \equiv \sigma(t)$ and $W_{\mu\nu} = 0.$

GR is immediately recovered if $\sigma(\Psi^i) = 1$ and $W_{\mu\nu} = 0$

Consider a perfect fluid $\longrightarrow$ $T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u_\mu u_\nu$

The extended Einstein field equations can also be written as [4]

$$G_{\mu\nu} = \kappa T^{\text{eff}}_{\mu\nu} = \frac{\kappa}{\sigma} T_{\mu\nu} + W_{\mu\nu}. \qquad (1)$$

We write conservation of the matter energy-momentum tensor: $\nabla_\mu T^{\mu\nu}_{\text{eff}} = 0.$ (1')

Consider static and spherically symmetric geometry is given by the metric, which is given:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \qquad (2)$$

As the metric is time independent and spherically symmetric, the pressure p and energy density ϱ are functions of the radial coordinate r only. Hence we'll assume that the coupling function σ and the geometric contributions W are also independent of the coordinates (t,θ,ϕ).

We calculate in detail the components of (1). The components of the Ricci tensor read [1]

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{B} + \frac{B'}{B}\right) - \frac{B'}{rA}$$

$$= \frac{\kappa}{2\sigma}(\rho + 3p)B + W_{tt} + \frac{BW}{2},$$

$$R_{rr} = \frac{B''}{2A} - \frac{B'}{4B}\left(\frac{A'}{B} + \frac{B'}{B}\right) - \frac{A'}{rA}$$

$$= \frac{\kappa}{2\sigma}(p - \rho)A + W_{rr} - \frac{AW}{2},$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{B} + \frac{B'}{B}\right) + \frac{1}{A}$$

$$= \frac{\kappa}{2\sigma}(p - \rho)r^2 + W_{\theta\theta} - \frac{r^2 W}{2}, \quad (3)$$

Where $W = -B^{-1}W_{tt} + A^{-1}W_{rr} + 2r^{-2}W_{\theta\theta}$ is the trace of the tensor. (‘)symbol donates the derivative respect to r. Using (3), we can write

$$\frac{R_{rr}}{2A} + \frac{R_{00}}{2B} + \frac{R_{\theta\theta}}{r^2} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = \frac{\kappa\rho}{\sigma} + r^2 B^{-1} W_{tt} \quad (4)$$

$$\left(\frac{r}{A}\right)' = 1 + \kappa r^2 \frac{\rho(r)}{\sigma(r)} + r^2 B^{-1}(r) W_{tt}(r). \quad (5)$$

Write the solution of (5)

$$A(r) = \left(1 - \frac{2G\mathcal{M}(r)}{r}\right)^{-1},$$

Where mass function:

$$\mathcal{M}(r) = \int_0^r \left(4\pi\tilde{r}^2 \frac{\rho(\tilde{r})}{\sigma(\tilde{r})} - \frac{\tilde{r}^2 W_{tt}(\tilde{r})}{2GB(\tilde{r})}\right) d\tilde{r}. \quad (6)$$

- This solution is clearly different from the usual definition given by GR in which $\mathcal{M}(R)$ is interpreted as the physical mass of the central object. Here, this expression should be interpreted as the mass function of the coupled TOV-like system. Geometric quantities also enter here. This expression is different from the actual physical mass.

The complete derivations also needs the relations

$$\frac{A'}{A} = \frac{1-A}{r} - \frac{\kappa A r}{\sigma} Q, \qquad (7)$$

$$\frac{B'}{B} = \frac{A-1}{r} - \frac{\kappa A r}{\sigma} \Pi,$$

where we've defined the new quantities:

$$Q(r) := \rho(r) + \frac{\sigma(r) W_{tt}(r)}{\kappa B(r)}, \qquad (8)$$

$$\Pi(r) := p(r) + \frac{\sigma(r) W_{rr}(r)}{\kappa A(r)}.$$

In GR the functions Q and Π would be interpreted as the energy density and pressure, respectively.
In hydrostatic equilibrium, from (1')

$$\kappa(\sigma^{-1}\nabla_\mu T^{\mu\nu} - \sigma^{-2} T^{\mu\nu}\nabla_\mu \sigma) + \nabla_\mu W^{\mu\nu} = 0, \qquad (9)$$

$$\kappa\sigma^{-1}\left(p' + (p+\rho)\frac{B'}{2B}\right) - \kappa p \frac{\sigma'}{\sigma^2} - \frac{A'}{A^2} W_{rr} + A^{-1} W_{rr}' + \frac{2W_{rr}}{Ar} + \frac{B'}{2B}\left(\frac{W_{rr}}{A} + \frac{W_{tt}}{B}\right) - \frac{2W_{\theta\theta}}{r^2} = 0. \qquad (10)$$

From (8), (7)

$$\left(\frac{\Pi}{\sigma}\right)' = \frac{p'}{\sigma} - \frac{p\sigma'}{\sigma^2} + \frac{W_{rr}'}{\kappa A} - \frac{W_{rr}(1-A)}{\kappa r A} + \frac{r W_{rr} Q}{\sigma}. \qquad (11)$$

This equation (11) is the basic structure for deriving the generalized hydrostatic equilibrium for stars in ETG. Together with (7) and definition (6), Eq. (11) can be written as

$$\left(\frac{\Pi}{\sigma}\right)' = -\frac{Gm}{r^2}\left(\frac{Q}{\sigma}+\frac{\Pi}{\sigma}\right)\left(1+\frac{4\pi r^3 \frac{\Pi}{\sigma}}{\mathcal{M}}\right) \times \left(1-\frac{2G\mathcal{M}}{r}\right)^{-1} + \frac{2\sigma}{\kappa r}\left(\frac{W_{\theta\theta}}{r^2}-\frac{W_{rr}}{A}\right). \quad (12)$$

and

$$\mathcal{M}(r) = \int_0^r 4\pi\tilde{r}^2 \frac{Q(\tilde{r})}{\sigma(\tilde{r})} \mathrm{d}\tilde{r} \quad (13)$$

(13) have the similar functional form to the standard GR result. But one remarkable difference is the existence of the geometrical contribution in the last term of (12). The set of equations (12) and (13) represent a useful tool for studying stellar configurations once a certain ETG is specified.

It should be noticed that since the tensor $W_{\mu\nu}$ can include some extra fields like scalar ones for example, besides the generalized Einstein's field equations (1), one will inevitably deal with equations of motion for the additional fields which should be taken into account. This means that Eqs. (12) and (13) are general up to the definition of the specific theory.

References

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