

The motion of a relativistic charged particle in the rotating pulsar magnetosphere

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The motion of one particle alongside pulsar magnetic field lines is considered. Because of pulsar's very strong magnetic field lines, we assume that the particle is on a zero Landau level and is not able to produce any synchrotron radiation. Based on these approximation we solve the bead-rod problem in relativistic case and find another solution for already well-known equation of motion of this system. We analyze different behaviors of particle's motion, which very much depends on initial parameters.

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Introduction: The motion of charged particles alongside pulsar magnetic field lines is a common problem in astrophysics, with many interesting variations. Numerous behaviours can be derived from different approaches of this fundamental problem. for example, one can get a vast variety of solutions for different shapes of magnetic field lines [7], or for taking friction into account[2] [3]. In several publications the same problem (particle motion along magnetic field lines) in different metrics (black hole[6], wormhole [4] etc. [5]) has been considered, and the main peculiarity of the corresponding solution is that one can get a negative centrifugal acceleration (force directed towards the rotational axis) at relativistic velocities [1]. It is also very popular to explain different mysterious emissions from Pulsars by centrifugally accelerated particles[8] [9] [10] [11].

In ref [1] a bead is considered to be inside very long pipe, which can rotate around its perpendicular axis (for simplicity we also consider rotation and magnetic field axis to be perpendicular). The bead is given initial speed very near to rotational axis and it turns out that when relativistic effect is considered, the bead performs oscillatory motion. Upon reaching the light cylinder, the radial velocity of the bead becomes zero and it starts to move towards rotational axis. The main peculiarity of the motion is that centrifugal force is directed towards the rotational axis. Of course, this model is partially unrealistic because of the fact that bead will never be able to oscillate, as no pipe is strong enough to hold a particle with almost infinite energy, but the approach and results still are spectacular as the origin of negative centrifugal force is manifested in simple relativistic system.

We associate this simple mechanical-relativistic problem with Pulsars. Pulsar's magnetic field is very strong and the effect of "freezing" particle to magnetic field lines is possible. Every particle that enters Pulsar's magnetic field, either from surface or from accretion disk, starts cyclonic motion around field line and immediately (af-

ter some pico seconds) goes to quantum (Landau) levels. This effect is called "freezing" to field lines. Because of this freezing, many scientists consider bead-rod problems. However, in the very first paper [1] that used this approach, authors only consider a particle to be at rotation axis and have some initial speed (which they vary and observe the results). In real astrophysical situations, usually a particle is influenced by Pulsar magnetic field when it leaves a thick layer of plasma on the surface. So, in real astrophysical scenarios the conditions of the relativistic bead-rod problem are as follows: initial distance from the rotational axis is not zero, but approximately the radius of Pulsar. This might be negligible for Normal Pulsars as their radius is several kilometers and the radius of light cylinder is $3 \cdot 10^5$ km (if we assume that angular frequency is $\omega = 1$). But for example as for millisecond Pulsars (ω can be up to 3000) the light cylinder is much closer and this distance becomes comparable to Pulsar's radius. Another possibility is that if we have a binary system and accretion disk is formed around Pulsar, particles can start "falling" on Pulsar surface. In this case one might get a scenario when a particle will not be able to reach the surface because it will get decelerated by centrifugal force, etc.

Let us consider the motion in normal Minkowskian metric (in units of $c = 1$):

$$ds^2 = -d\tau^2 = -dT^2 + dX^2 + dY^2 \quad (1)$$

Making the following transformation of variables (for simplicity, also considering that rotational axis is perpendicular to XY plane):

$$T = t \quad X = r \cos(\omega t) \quad Y = r \sin(\omega t) \quad (2)$$

The two-dimensional metric in the rotating frame will be the following:

$$-d\tau^2 = -(1 - \omega^2 r^2) dt^2 + dr^2 \quad (3)$$

where ω is rotational angular frequency and r is distance from the rotational axis. Thus the Lagrangian reads as:

$$L = \frac{1}{2} \left[- (1 - \omega^2 r^2) \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{dr}{d\tau} \right)^2 \right] \quad (4)$$

From (4), we can write Lagrange equation:

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{\partial L}{\partial x^\alpha} \quad (5)$$

And given Lagrange equation can be written:

$$- (1 - \omega^2 r^2) \left(\frac{dt}{d\tau} \right) = \text{const} \equiv -E \quad (6)$$

$$\frac{d^2 r}{d\tau^2} = \omega^2 r \left(\frac{dt}{d\tau} \right)^2 \quad (7)$$

From (6), (7) and also by using the metric (3) we can get governing equation of motion for the bead.

$$\frac{dr}{dt} = \sqrt{(1 - \omega^2 r^2) \left(1 - \frac{(1 - \omega^2 r^2)}{E} \right)} \quad (8)$$

where E is invariant and only depends on initial conditions as follows: according to (8), at $t = 0$, let us denote that $\frac{dr}{dt} = v_0$ and $r = r_0$, thus E will be:

$$E = \frac{1 - \omega^2 r_0^2}{\sqrt{1 - \omega^2 r_0^2 - v_0^2}} \quad (9)$$

The character of the solution of this equation depends on the value of E . If $E > 1$ the solution will be elliptic cosine that was derived originally in Ref. [1]. If $E < 1$ solution will be different (but see below). In order to find the solutions of equation (8), we should define the variables θ , λ and m in the following way:

$$\theta = \text{acos}(\omega r) \quad T = \omega t \quad m = \frac{1}{E^2} \quad (10)$$

And get a simplified equation:

$$\frac{d\theta}{dT} = -\sqrt{1 - m \sin^2 \theta} \quad (11)$$

Which is the definition of elliptic cosine (cn function), if $m < 1$ and thus:

$$r(t) = \frac{1}{\omega} \text{cn}(\lambda - \omega t, m) \quad (12)$$

where λ is incomplete elliptic integral of the first kind:

$$\lambda = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \quad (13)$$

where $\phi = \text{acos}(\omega r)$, $m = 1/E^2$ note that m should be less than 1, otherwise given integral will be undefined.

from equation (12), it is clear that the motion will be oscillatory, the particle will have zero radial speed when it reaches the light cylinder and it will start moving towards rotation axis after that point. In real systems, this oscillations will not happen, because the magnetic field lines will not be able to hold a particle with infinite energy and thus its tangential velocity ωr will not reach the speed of light. Before reaching light cylinder, particles will start bending magnetic field lines and at some point they will not be influenced by them at all.

As we mentioned earlier, (12) does not describe the motion if initial energy $E < 1$ (m is proportional to $\frac{1}{E^2}$, so when $E < 1$, elliptical integral according to eq. (13) is undefined).

In order to get analytical solutions for Eq. (8), in case when $E > 1$, we denote $m = E^2$ and the solution for Eq. (8) reads as:

$$r(t) = \frac{1}{\omega} \text{dn} \left(\frac{\omega}{E} t - \lambda, m \right) \quad (14)$$

In this case, λ is the same as in Eq. (13), but with different $\phi = \text{asin} \sqrt{\frac{1 - \omega^2 r_0^2}{m}}$ and $m = E^2$.

If we look at invariant Energy which only depends on initial conditions in Eq. (9), we can physically determine which analytical solution (Eq. (12) or Eq.(14)) should be used. If initial radial velocity is zero, energy (E) will always be more than one (unless $r_0 = 0$, which will mean that particle is on the rotational axis and does not have any velocity. This is unstable trivial solution), so the solution for this initial conditions will be Eq. (14). On the other hand, if particle is initially located on the rotational axis ($r_0 = 0$), the solution will be Eq. (12).

And in real physical case, when particle starts its centrifugal motion with some given velocity and also the distance to rotational axis is not negligible, only one from Eqs. (12) and (14) will be the solution to this original problem.

Results: After deriving generalized equations of motion for any initial parameters (elliptic functions cn and dn), now we will study particle's behavior for different initial conditions.

Figures 1. 2. and 3. show us respectively the graphs (for $\omega = 1$) of radial distance dependence on time, radial velocity dependence on time and radial acceleration dependence on time. In every Figure a) and b) subfigures represent solutions for elliptic cosine. c) and d) subfigures represent solutions for elliptic function dn.

In FIG. 1 one can see particle's radial distance (from rotational axis) dependence on time. subfigure a) shows that the solution will be elliptic cosine if in the beginning particle is on the rotational axis and has some initial velocity, and reversely, if initially particle is far from rotational axis and has velocity equal to zero (subfigure c),

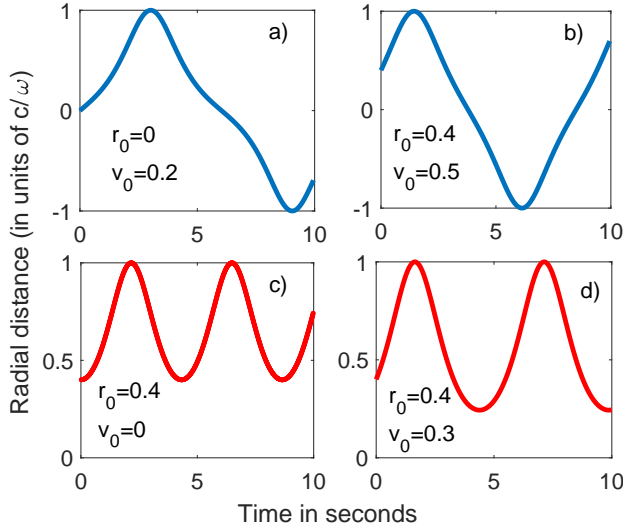


FIG. 1: These figures represent dependence of particle's distance from rotational axis on time. a) and b) are graphs of cn function. c). and d) are for dn function. Initial conditions is given on each graph, and for all cases Rotator's (Pulsar's) rotational angular frequency is $\omega = 1$

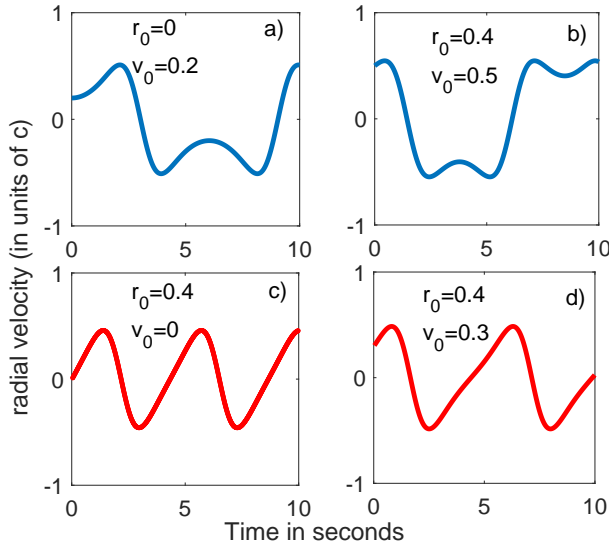


FIG. 2: These figures represent dependence of particle's absolute velocity on time (we have plotted absolute velocity because the orientation of particle's motion is easily determinable from both these and previous graphs). a) and b) are graphs of cn function. c). and d) are for dn function. Initial conditions is given on each graph (and are the same as in the previous figure), and for all cases Rotator's (Pulsar's) rotational angular frequency is $\omega = 1$

its motion will be elliptic function dn. However, if initially distance from rotational axis and velocity both are nonzero, the solution will be either cn or dn. if velocity is large enough for the particle to reach rotational axis with nonzero speed, then the solution will be cn (subfigure b),

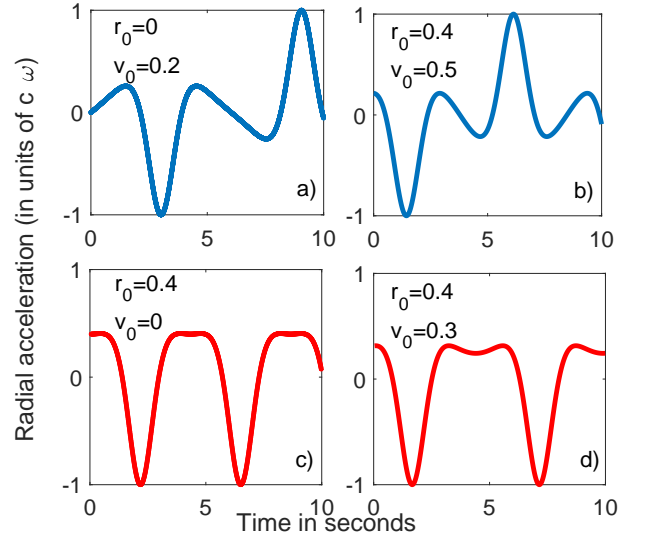


FIG. 3: These figures represent dependence of acceleration on time. a) and b) are graphs of cn function. c). and d) are for dn function. Initial conditions is given on each graph (and are the same as in previous figures), and for all cases Rotator's (Pulsar's) rotational angular frequency is $\omega = 1$

and otherwise (if the particle will not be able to reach rotational axis and will stop somewhere) the solution will be dn. Also, we can distinguish these two motions (cn and dn) easily: it is cn, if particle oscillates in the range between -1 and 1. it is dn, if particle oscillates in the range between 0 and 1 (the upper boundary should be 1. However, lower limit is not fixed and can be anything from 0 to 1, it depends on initial conditions).

Figure 2. shows particle's absolute radial velocity dependence on time for the same initial conditions as in FIG. 1.

One of the main peculiarities of this problem is that acceleration is not always positive and particle can be attracted towards rotational axis (negative value of acceleration is only at relativistic velocities). In Fig. 3 one can observe particle's acceleration dependence on time, for the same parameters as in Fig. 1. We should also note that acceleration can even be negative at the very beginning of motion, if the initial kinetic energy is too large. For example, if $r_0 = 0$ and $v_0 > \frac{\sqrt{g r t^2}}{2}$, acceleration will be negative at $t = 0$. However, if $v_0 = 0$ acceleration can not be negative for any initial distance r_0 .

All the figures have been plotted for $\omega = 1$ (rotational angular frequency). We are suggesting that in many real physical systems, such as millisecond Pulsars, one would need to use the solution of elliptic function dn, because millisecond Pulsars have very small Period (thus very large angular frequency, up to 1000-3000 Hz), and even Pulsar's radius is comparable to the length of its light cylinder. Thus, questions may arise why we only plotted graphs for $\omega = 1$: That is because, changing ω does

not change the characteristics of particle's motion. The shape of graphs remain the same, only change is in time needed to reach the light cylinder, the absolute values of velocities and accelerations, but the shape of the Plot is the same and there is nothing interesting except numbers.

Also, we did not Plot the dependence of Lorentz factor on time or any other parameter. That is because of the fact that during the motion, Lorentz factor increases and tends to infinity near light cylinder for any initial parameter. That is why in real physical systems no magnetic field will be able to hold the particle on Landau levels near light cylinder. As particle's mass grows, it will be less and less affected by magnetic field and after some point the particle will be moving linearly.

Discussions and Conclusion: We have mentioned many times that this is the idealized problem, not because of the fact that we consider particle "frozen" to the magnetic field lines (this is very realistic for very strong magnetic fields, such as near Pulsars, particles are on Landau levels and are not able to emit any synchrotron radiation), but because at light cylinder particle's total velocity becomes equal to speed of light (not for finite time) and thus the mass increases rapidly as the particle gets closer to the light cylinder. So, magnetic field itself has to be infinitely large to hold a particle on Landau levels and accelerate it centrifugally. So, we only say that for realistic systems our equations are only valid before the particle gets very close to light cylinder.

In fact, to consider a problem of particle's behavior near light cylinder would be very interesting (and we plan to do it in our future works). Many behaviors may occur, for example particle (which will be very massive) can "escape" from Landau levels and emit synchrotron radiation, or we might find a point from where on magnetic field's influence might become negligible and the motion could become linear.

One can also consider radiation (because of accelerated motion) from particle's centrifugal motion described in this work and compare its spectra to real physical objects.

In conclusion, we would like to say that we have found another very important solution for this original problem and alongside with highlighting the results we discussed in which physical scenarios one can use the solution that is represented by elliptical function dn . In particular, it can be used to describe particle's motion alongside millisecond Pulsar's straight magnetic field lines, because Pulsar radius is comparable to light cylinder diameter and in many cases (it depends on initial velocity) second solution (elliptical function dn) would be needed to describe the motion.

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